4/10/03

A cohomology operation of degree i is a family of natural transformations

$$H^n(X,A;k) \xrightarrow{\Theta} H^{n+i}(X,A;k).$$

We say that Θ is <u>stable</u> if

$$\begin{tikzcd} H^n(A;k) \arrow[r, "\Theta"] \arrow[d, "\partial"] & H^{n+i}(A;k) \arrow[d, "\partial"] \\ H^{n+1}(X,A;k) \arrow[r, "\Theta"] & H^{n+1+i}(X,A;k) \end{tikzcd}$$

commutes, for all pairs (X,A) and all $n \geq 0$. The sum of two stable coh. op. is a stable coh. op.; the composition of two stable coh. op. is a stable coh. op.; mult. by $a \in k$ is a stable coh. op. of degree zero. It follows that the collection of all stable coh. op. form a graded k-algebra $\mathcal{A}_k^*$, the Steenrod algebra. There is no reason that $\mathcal{A}_k^*$ should be a commutative k-algebra, and it is not (unless k is a $\mathbb{Q}$-algebra). Today, we consider

$$k = \mathbb{F}_2 .$$

The Steenrod squares are stable coh. op.

$$H^n(X) \xrightarrow{Sq^i} H^{n+i}(X)$$

which we now construct. We let

$$F(X,Y) = C_*(X \times Y)$$
$$G(X,Y) = C_*(X) \otimes C_*(Y),$$

viewed as functors from pairs of spaces to graded k-modules (forget the diff.), and define $\mathrm{Hom}(F,G)_r$ to be the k-module of nat. transf. of degree r from F to G, if $r \geq 0$, and to be zero, if $r < 0$. We define a diff.

$$\mathrm{Hom}(F,G)_r \xrightarrow{d} \mathrm{Hom}(F,G)_{r-1}$$

by the equation

$$d(f(x)) = (df)(x) + (-1)^{|f|} f(dx).$$

Thm There is a canonical quasi-isom.

$$\mathrm{Hom}(F,G) \xrightarrow[\sim]{\varepsilon} k.$$

//

We define an action by the symmetric

group $\Sigma_2 = C_2 = \langle T \rangle$ on $\mathrm{Hom}(F, G)$ as follows. We have chain maps

$$F(X, Y) \xrightarrow{T_*} F(Y, X)$$

$$G(X, Y) \xrightarrow{T_*} G(Y, X)$$

given by the two twist maps, and def.

$$\mathrm{Hom}(F, G) \xrightarrow{T} \mathrm{Hom}(F, G)$$

$$Tf = T_* f T_*^{-1}.$$

Hence, the Eilenberg-Zilber theorem gives a resolution of k by kC_2 modules. We also have the standard resolution

$$W \underset{\sim}{\xrightarrow{\varepsilon}} k,$$

where $W_i = k[C_2] \cdot e_i$, where

$$d e_i = \begin{cases} (1+T)\, e_{i-1}, & i \text{ even}, \\ (1-T)\, e_{i-1}, & i \text{ odd}, \end{cases}$$

and where $\varepsilon(e_0) = 1$. By the fund. theorem of homological algebra, there exists a chain map of cx. of

kC_2 - modules

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[d, "\varepsilon"] & \mathrm{Hom}(F, G) \arrow[d, "\varepsilon"] \\ k \arrow[r, equal] & k \end{tikzcd},$$

and any two such maps are chain homotopic. From $a = \alpha(e_0)$ we get the chain map

$$C_*(X \times Y) \xrightarrow{a} C_*(X) \otimes C_*(Y)$$

that we used to define the cup product. We shall now use α in a similar manner. From α, we get a chain map of cx. of $k[C_2]$-modules

$$W \otimes C_*(X \times X) \xrightarrow{\alpha_*} C_*(X) \otimes C_*(X),$$

and dually,

$$W \otimes (C^*(X) \otimes C^*(X)) \xrightarrow{\alpha^*} C^*(X \times X).$$

Composing the latter with

$$C^*(X \times X) \xrightarrow{\Delta^*} C^*(X),$$

we get a map of cx. of kC_2-mod.

$$W \otimes (C^*(X) \otimes C^*(X)) \xrightarrow{\tilde{\varphi}} C^*(X).$$

But C_2 acts trivially on the right, so this induces a chain map of cx. of k-mod.

$$W \otimes_{kC_2} (C^*(X) \otimes C^*(X)) \xrightarrow{\varphi} C^*(X),$$

i.e.

$$\varphi(T.e_i \otimes (x \otimes y)) = \varphi(e_i \otimes (y \otimes x)).$$

We define higher cup products

$$C^*(X) \otimes C^*(Y) \xrightarrow{\cup_i} C^{*-i}(X)$$

$$x \otimes y \longmapsto \varphi(e_i \otimes (x \otimes y)).$$

Here $\cup_0 = \cup$ is a chain map and induces the cup prod. on coh. But $\cup_i$, $i>0$, are not chain maps. Instead we have:

$$d(x \cup_i y) = d\varphi(e_i \otimes (x \otimes y))$$

$$= \varphi(d(e_i \otimes (x \otimes y)))$$

$$= \varphi\big((e_{i-1} + Te_{i-1}) \otimes (x \otimes y)\big)$$

$$+ \ \varphi\big(e_i \otimes (dx \otimes y + x \otimes dy)\big)$$

$$= x \cup_{i-1} y + y \cup_{i-1} x + dx \cup_i y + x \cup_i dy .$$

Here we use that k has char. 2. We now define

$$C^n(X) \xrightarrow{\ sq^i\ } C^{n+i}(X)$$

$$x \longmapsto x \cup_{n-i} x + x \cup_{n-i+1} dx$$

and calc.

$$d(sq^i x) = d(x \cup_{n-i} x + x \cup_{n-i+1} dx)$$

$$= \cancel{x \cup_{n-i-1} x} + \cancel{x \cup_{n-i-1} x} + \cancel{dx \cup_{n-i} x} + \cancel{x \cup_{n-i} dx}$$

$$+ \cancel{x \cup_{n-i} dx} + \cancel{dx \cup_{n-i} x} + dx \cup_{n-i+1} dx$$

$$= sq^i(dx) .$$

It is not clear that sq^i is additive, and in fact, it is not:

$$sq^i(x+y) = (x+y) \cup_{n-i} (x+y)$$

$$+ (x+y) \cup_{n-i+1} d(x+y)$$

$$= x \cup_{n-i} x + x \cup_{n-i} y + y \cup_{n-i} x + y \cup_{n-i} y$$

$$+ x \cup_{n-i+1} dx + x \cup_{n-i+1} dy + y \cup_{n-i+1} dx$$

$$+ y \cup_{n-i+1} dy$$

$$= Sq^i(x) + Sq^i(y) + d(x \cup_{n-i+1} y)$$

$$+ dx \cup_{n-i+1} y + y \cup_{n-i+1} dx$$

so if x and y are cycles,

$$Sq^i(x+y) = Sq^i(x) + Sq^i(y) + d(x \cup_{n-i+1} y).$$

Hence, we have an additive map

$$\begin{aligned} H^*(X) &\xrightarrow{Sq^i} H^{*+i}(X) \\ [x] &\longmapsto [Sq^i(x)] . \end{aligned}$$

The formula for $Sq^i(x+y)$ also shows

$$\begin{tikzcd} 0 \arrow[r] & C^n(A) \arrow[r] \arrow[d, "Sq^i"] & C^n(X) \arrow[r] \arrow[d, "Sq^i"] & C^n(X,A) \arrow[r] \arrow[d, dashed, "Sq^i"] & 0 \\ 0 \arrow[r] & C^{n+i}(A) \arrow[r] & C^{n+i}(X) \arrow[r] & C^{n+i}(X,A) \arrow[r] & 0 . \end{tikzcd}$$

Easy to check that Sq^i is stable.

Lemma (i) $Sq^n(x) = x \cup x$ if $|x| = n$.

(ii) $Sq^i \neq 0$ for all $i \geq 0$.

pf (i) $Sq^i(x) = x \cup_0 x + x \cup_1 dx = x \cup x$.

(ii) $H^*(\mathbb{RP}^\infty) = S_{\mathbb{F}_2}\{x\}$, $|x| = 1$, so

$$Sq^i(x^i) = x^i \cup x^i = x^{2i} \neq 0. \qquad //$$

Rem It is not clear that $Sq^i = 0$, $i < 0$.

$$\boxed{\begin{aligned} Sq^{2n-1}_{*}\, Sq^{n}_{*} &= 0 \\ \chi\,(Sq^{n}_{*}) &= Sq^{n-1}_{*} \end{aligned}}$$

Prop Let X be a conn. CW-cx. Then for all $n \geq 0$, there exist a CW-cx $\tau_{\leq n} X$ and a cellular map

$$X \xrightarrow{f} \tau_{\leq n} X$$

s.t. $\pi_i(\tau_{\leq n} X) = 0$, $i > n$, and s.t. $\pi_i(f)$ is an iso., $i \leq n$.

pf It clearly suffices to construct a cellular map

$$X \xrightarrow{g} Y$$

s.t. $\pi_i(g)$ is an iso, $i \leq n$, and s.t. $\pi_{n+1}(Y) = 0$. Let $S^{n+1} \xrightarrow{\varphi_\alpha} X$, $\alpha \in A$, be a set of cellular maps that generate $\pi_{n+1}(X)$ and define

$$\begin{tikzcd}
\bigvee_{\alpha \in A} S^{n+1} \arrow[r, "\varphi_\alpha"] \arrow[d, hook] \arrow[dr, phantom, "p.o."] & X \arrow[d, hook, "g"] \\
\bigvee_{\alpha \in A} D^{n+2} \arrow[r] & Y
\end{tikzcd}$$

Then Y is a CW-cx. and g is the inclusion of the $(n+1)$-skeleton. Hence g is $(n+1)$-connected, so $\pi_i(g)$ is an iso, $i \leq n$. By cellular approx., every map

$$S^{n+1} \xrightarrow{f} Y$$

is homotopic to a (cellular) map

$$S^{n+1} \xrightarrow{f'} X \hookrightarrow Y.$$

But every such map is null-htpic by construction. //

Cor Let A be an abelian group, and let $n \geq 2$. Then there exist a CW-cx. X whose $(n-1)$-skeleton is a point and such that

$$\pi_i(X) \approx \begin{cases} A & , i = n, \\ 0 & , i \neq n. \end{cases}$$

pf We pick a free presentation

$$0 \to \bigoplus_{j \in J} \mathbb{Z} \xrightarrow{f} \bigoplus_{i \in I} \mathbb{Z} \to A \to 0$$

and consider the mapping cone

$$\bigvee_{j \in J} S^n \xrightarrow{f} \bigvee_{i \in I} S^n \longrightarrow C_f .$$

Clearly,

$$\tilde{H}_i(C_f, \mathbb{Z}) \approx \begin{cases} A & , i = n, \\ 0 & , \text{else}, \end{cases}$$

so by Hurewicz,

$$\pi_i(C_f) \approx \begin{cases} A & , i = n, \\ 0 & , i < n. \end{cases}$$

Hence, $X = \tau_{\leq n} C_f$ does the job. //

A connected space X s.t.

$$\pi_i(X) \approx \begin{cases} A & , i = n, \\ 0 & , i \neq n, \end{cases}$$

is called an Eilenberg–MacLane space of type (A, n). The weak htpy. of such a space is unique. Indeed:

<u>Lemma</u> Let X and X' be Eilenberg-MacLane spaces of type (A, n), and suppose that X is a CW-cx. Then there exists a w-e.

$$X \xrightarrow{\sim} X'.$$

<u>Pf</u> We can assume that X is the CW-cx. that we constructed above:

$$\bigvee_{j \in J} S^n \xrightarrow{f} \bigvee_{i \in I} S^n \longrightarrow C_f,$$

$$X = \tau_{\leq n} C_f.$$

We choose maps

$$S^n \xrightarrow{\varphi_i} X'$$

that repr. the gen. of $\pi_n(X') \approx A$ corresponding to $i \in I$, and null homotopies

$$D^{n+1} \xrightarrow{h_j} X'$$

of the composites

$$S^n \xrightarrow{\iota_j} \bigvee_{j\in J} S^n \xrightarrow{f} \bigvee_{i\in I} S^n \xrightarrow{\vee \varphi_i} X'$$

This gives a map

$$\begin{tikzcd}
\bigvee_{j\in J} S^n \arrow[r, "f"] \arrow[d, hook] \arrow[dr, phantom, "\text{p.o.}"] & \bigvee_{i\in I} S^n \arrow[d, hook] \arrow[ddr, bend left, "\vee \varphi_i"] & \\
\bigvee_{j\in J} D^{n+1} \arrow[r] \arrow[drr, bend right, "\vee h_j"'] & C_f \arrow[dr, "g"] & \\
& & X'
\end{tikzcd}$$

s.t. $\pi_n(g)$ is an iso. Since $\pi_i(X')=0$, $i>n$, this extends to a map

$$X \xrightarrow{g} X'$$

s.t. $\pi_i(g)$ is an iso, for all i. //

Cor Let $K(A,n)$ be an Eilenberg-MacLane cx. of type (A,n), $n\geq 0$. Then there are weak equivalences

$$K(A,n) \xrightarrow{\sim} \Omega K(A,n+1).$$

//

We choose (pointed) Eilenberg-MacLane cx. $K(A,n)$, $n \geq 0$, and (pointed) w.e.

$$K(A,n) \xrightarrow[\sim]{\sigma_n} \Omega K(A,n+1).$$

If X is a pointed CW-cx., we define

$$\tilde{h}^n(X;A) := [X, K(A,n)]$$

to be the set of base-point preserving homotopy classes of base-point preserving maps. It is an abelian group, since

$$K(A,n) \xrightarrow{\sim} \Omega^2 K(A,n+2)$$

is a homotopy commutative H-space. Let $Y \overset{i}{\hookrightarrow} X$ be the inclusion of a pointed sub-cx. and define

$$h^n(X,Y;A) := \tilde{h}^n(C_i;A).$$

The sequence of maps

$$Y \overset{i}{\hookrightarrow} X \overset{j}{\hookrightarrow} C_i \xrightarrow{p} \Sigma Y$$

induces an exact sequence

that converges to $\tilde{h}^*(X;A)$ and to $\tilde{H}^*(X;A)$, respectively. Both have

$$E_2^{s,t} = \begin{cases} \tilde{H}^s_{CW}(X;A) & , t=0, \\ 0 & , t>0. \end{cases}$$ //

—

Let $E \xrightarrow{f} B$ be a Serre fibration with B simply-connected and with fiber $F = F_b$. Consider the maps

$$H^{r-1}(F) \xrightarrow{\partial} H^r(E,F) \xleftarrow{f^*} H^r(B,\{b\}) \xrightarrow{\sim} H^r(B).$$

and the Serre sp. seq.

$$E_2^{s,t} = H^s(B, H^t(F)) \Rightarrow H^{s+t}(E).$$

Prop Let $x \in H^{r-1}(F) = E_2^{0,r-1}$ and suppose that there exists $y \in H^r(B) = E_2^{r,0}$ s.t. $f^*(y) = \partial(x)$. Then $d_i(x) = 0$, $i<r$, and

$$E_r^{0,r-1} \xrightarrow{d_r} E_r^{r,0}$$

maps x to the class of y. //

An elem. $x \in E_2^{0,r-1}$ s.t. $d_i(x) = 0$, $i < r$, is said to be <u>transgressive</u>. Let $H^*(-)$ be $H^*(-, \mathbb{F}_2)$ and recall the Steenrod sq.

$$H^n(X) \xrightarrow{Sq^i} H^{n+i}(X).$$

<u>Cor</u> Suppose that $x \in H^{r-1}(F) = E_2^{0,r-1}$ is transgressive and that $y \in H^r(B) = E_2^{r,0}$ repr. $d_r(x)$. Then $Sq^i(x) \in H^{r+i-1}(F) = E_2^{0,r+i-1}$ is transgressive, and $Sq^i(y) \in H^{r+i}(B) = E_2^{r+i,0}$ repr. $d_{r+i}(Sq^i(x))$.

<u>pf</u> This follows from the prop. and

$$\begin{tikzcd}
H^{r-1}(F) \arrow[r, "\partial"] \arrow[d, "Sq^i"] & H^r(E,F) \arrow[d, "Sq^i"] & H^r(B,\{b\}) \arrow[l, "f^*"'] \arrow[r, "\sim"] \arrow[d, "Sq^i"] & H^r(B) \arrow[d, "Sq^i"] \\
H^{r+i-1}(F) \arrow[r, "\partial"] & H^{r+i}(E,F) & H^{r+i}(B,\{b\}) \arrow[l, "f^*"'] \arrow[r, "\sim"] & H^{r+i}(B)
\end{tikzcd}$$

//

$$\tilde{h}^n(Y;A) \xleftarrow{i^*} \tilde{h}^n(X;A) \xleftarrow{j^*} \tilde{h}^n(X,Y;A) \xleftarrow{\phi^*} \tilde{h}^n(\Sigma\ ;A)$$

and the w.e. σ_{n-1} gives an iso.

$$\tilde{h}^n(\Sigma Y;A) = [\Sigma Y, K(A,n)]$$

$$\approx [Y, \Omega K(A,n)] \underset{\sim}{\xleftarrow{\sigma_{n-1}}} \tilde{h}^{n-1}(Y;A).$$

So we have a l.e.s. assoc. with the pair (X,Y)

$$\cdots \to \tilde{h}^{n-1}(Y;A) \xrightarrow{\partial} \tilde{h}^n(X,Y;A) \to \tilde{h}^n(X;A) \to \cdots$$

Moreover, we have canonical iso.

$$\tilde{h}^n(\bigvee_\alpha X_\alpha;A) \xrightarrow{\sim} \prod_\alpha \tilde{h}^n(X_\alpha;A).$$

Prop If X is a pointed CW-cx. then

$$\tilde{h}^n(X;A) \approx \tilde{H}^n(X;A).$$

pf The skeleton filtration of X gives rise to two natural spectral sequences

If $I = (i_1, \ldots, i_r)$, define

$$Sq^I = Sq^{i_1} \cdots Sq^{i_r} \qquad (Sq^{\emptyset} = \mathrm{id}).$$

Call I <u>admissable</u> if $i_m \geq 2 i_{m+1}$; define the <u>excess</u> of I by

$$\begin{aligned} e(I) &= (i_1 - 2i_2) + \cdots + (i_{r-1} - 2i_r) + i_r \\ &= i_1 - (i_2 + \cdots + i_r) \\ &= 2i_1 - \underbrace{(i_1 + \cdots + i_r)}_{n(I)} \end{aligned}$$

<u>ex</u> $e(I) = 0 \Rightarrow I = \emptyset \Rightarrow Sq^I = 1$.

$e(I) = 1 \Rightarrow I = (2^{r-1}, \ldots, 2, 1)$, $r \geq 0 \Rightarrow$

$$Sq^I = Sq^{2^{r-1}} \cdots Sq^2 Sq^1, \quad r \geq 0.$$

<u>Thm</u> (Serre) Let $\iota_n \in H^n(K(\mathbb{Z}/2\mathbb{Z}, n))$ be the unique generator. Then, as a ring,

$$H^*(K(\mathbb{Z}/2\mathbb{Z}, n)) = S_{\mathbb{F}_2}\{Sq^I \iota_n \mid I \text{ adm.}, e(I) < n\}.$$

<u>Pf</u> By induction on $n \geq 1$; $n = 1$ O.K.

So assume the statement for $n-1$; we consider the Serre sp. seq. assoc. w.

$$K(\mathbb{Z}/2\mathbb{Z}, n-1) \to PK(\mathbb{Z}/2\mathbb{Z}, n) \to K(\mathbb{Z}/2\mathbb{Z}, n).$$

The class $\iota_{n-1} \in E_2^{0,n-1}$ is transgressive for degree reasons, and $d_n(\iota_{n-1}) = \iota_n \in E_2^{n,0}$. By the transgression theorem, $Sq^I(\iota_{n-1})$ is transgressive (for any I), and

$$d_{n(I)+n}(Sq^I \iota_{n-1}) = Sq^I \iota_n .$$

We note that a basis of $H^*(K(\mathbb{Z}/2\mathbb{Z}, n-1))$ as (graded) $\mathbb{F}_2$-v.s. is given by all possible products of distinct classes of the form

$$(Sq^J \iota_{n-1})^{2^s} = Sq^{2^{s-1}a} \cdots Sq^{2a}\, Sq^{a}\, Sq^{J} \iota_{n-1},$$

with J adm., $e(J) < n-1$, and $s \geq 0$. Here $a = n(J) + n - 1$. Abstract sp. seq:

$${}'E_2^{0,*} = S_{\mathbb{F}_2}\{ Sq^J \iota_{n-1} \mid J \text{ adm.}, e(J) < n-1 \}$$

$${}'E_2^{*,0} = S_{\mathbb{F}_2}\{ Sq^{2^{s-1}a} \cdots Sq^a Sq^J \iota_n \mid J \text{ adm.}, e(J) < n-1, \ s \geq 0 \}$$

$'E_2^{*,*} = 'E_2^{*,0} \otimes 'E_2^{0,*}$ ⇝ calc. $E_r^{\prime *,*}$ over ⇝

with differentials the unique derivations determined by

$$'d\left(Sq^{2^{s-1}a} \cdots Sq^{2a} Sq^{a} Sq^{J} \iota_{n-1}\right)$$

$$= Sq^{2^{s-1}a} \cdots Sq^{2a} Sq^{a} Sq^{J} \iota_n \, .$$

The transgression theorem shows that this sp. seq. maps to the Serre sp. seq. The map is an iso. on $E_2^{0,*}$ and on $E_\infty^{*,*}$. By the comparison thm., it is an iso. on $E_2^{*,0}$. It follows that $H^*(K(\mathbb{Z}/2\mathbb{Z}, n))$ is a polynomial alg. over $\mathbb{F}_2$ gen. by the elem.

$$Sq^{2^{s-1}a} \cdots Sq^{2a} Sq^{a} Sq^{J} \iota_n$$

with J adm., $e(J) < n-1$, and $s \geq 0$. Write $J = (j_1, \ldots, j_r)$. The seq.

$$I = (2^{s-1}a, \ldots, 2a, a, j_1, \ldots, j_r) \, ,$$

where $a = j_1 + \cdots + j_r + n - 1$, is admissable:

$$a = n(J) + n - 1 \geq 2j_1 \iff$$

$$n-1 \geq 2j_1 - n(J) = e(J). \checkmark$$

The excess is

$$e(I) = \begin{cases} e(J) & , s = 0 \\ n-1 & , s > 0. \end{cases}$$

So I is adm. and $e(I) < n$. Conversely, suppose $I = (i_1, \dots, i_m)$ is adm., $e(I) < n$. Let $s \leq m$ be smallest s.t. $i_s > 2i_{s+1}$, let $a = i_s$, and let

$$J = (j_1, \dots, j_r) = (i_{s+1}, \dots, i_m).$$

Then $e(J) < n-1$. //

Cor As a graded $\mathbb{F}_2$-vector space

$$\mathcal{A}^*_{\mathbb{F}_2} = \mathbb{F}_2\{Sq^I \mid I \text{ adm.}\}.$$

pf By definition, $\mathcal{A}^*_{\mathbb{F}_2}$ is the (graded) algebra of stable coh. operations in $H^*(-, \mathbb{F}_2)$. The operations Sq^I are stable. The group $H^{n+i}(K(\mathbb{Z}/2, n))$ is isomorphic to the group of natural transformations

$$H^n(-, \mathbb{F}_2) \longrightarrow H^{n+i}(-, \mathbb{F}_2)$$

If $\theta = \{\theta_n\}$ is a stable op. of deg i, then θ_n is such a natural transformation. We have $\theta = 0$ if and only if $\theta_n = 0$, for all $n \geq 0$. Consider the map

$$\mathcal{A}^*_{\mathbb{F}_2} \longrightarrow H^{n+*}(K(\mathbb{Z}/2\mathbb{Z}, n))$$

$$\theta \longmapsto \theta(\iota_n)$$

takes θ to the elem. of the group on the right that corresponds to θ_n. The calc. above shows that a basis of $H^{n+i}(K(\mathbb{Z}/2\mathbb{Z}, n))$ with $i < n$ is given by the $Sq^I \iota_n$ with I adm. and $n(I) = i$. (Note that $n(I) < n$ implies $e(I) < n$.) Hence every natural transformation

$$H^n(-, \mathbb{F}_2) \longrightarrow H^{n+i}(-, \mathbb{F}_2)$$

with $i < n$ is induced from a stable operation. Letting n vary, we see that a basis of the stable op. is as stated. //

Let k be a commutative ring. A (graded) <u>Hopf algebra</u> over k is a (graded) k-module A together with maps of (graded) k-modules

$$A \otimes A \xrightarrow{\varphi} A$$

$$A \xrightarrow{\psi} A \otimes A$$

$$k \xrightarrow{\eta} A$$

$$A \xrightarrow{\varepsilon} k$$

$$A \xrightarrow{c} A$$

such that the following diagr. comm.

(i)
$$\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{A \otimes \varphi} & A \otimes A \\
\downarrow{\scriptstyle \varphi \otimes A} & & \downarrow{\scriptstyle \varphi} \\
A \otimes A & \xrightarrow{\varphi} & A
\end{array}$$

(ii)
$$\begin{array}{ccc}
A \otimes A \otimes A & \xleftarrow{A \otimes \psi} & A \otimes A \\
\uparrow{\scriptstyle \psi \otimes A} & & \uparrow{\scriptstyle \psi} \\
A \otimes A & \xleftarrow{\psi} & A
\end{array}$$

(iii)
$$
\begin{tikzcd}
k \otimes A \arrow[r, "\eta \otimes A"] \arrow[dr, "\sim"'] & A \otimes A \arrow[d, "\varphi"] & A \otimes k \arrow[l, "A \otimes \eta"'] \arrow[dl, "\sim"] \\
& A &
\end{tikzcd}
$$

(iv)
$$
\begin{tikzcd}
k \otimes A & A \otimes A \arrow[l, "\varepsilon \otimes A"'] \arrow[r, "A \otimes \varepsilon"] & A \otimes k \\
& A \arrow[ul, "\sim"] \arrow[u, "\psi"'] \arrow[ur, "\sim"'] &
\end{tikzcd}
$$

(v)
$$
\begin{tikzcd}
A \otimes A \arrow[r, "\varphi"] \arrow[d, "\psi \otimes \psi"] & A \arrow[r, "\psi"] & A \otimes A \\
A \otimes A \otimes A \otimes A \arrow[rr, "A \otimes tw \otimes A"] & & A \otimes A \otimes A \otimes A \arrow[u, "\varphi \otimes \varphi"']
\end{tikzcd}
$$

(vi)
$$
\begin{tikzcd}
A \otimes A \arrow[dd, "A \otimes c"] & A \arrow[l, "\psi"'] \arrow[r, "\psi"] \arrow[d, "\varepsilon"] & A \otimes A \arrow[dd, "c \otimes A"] \\
& k \arrow[d, "\eta"] & \\
A \otimes A \arrow[r, "\varphi"] & A & A \otimes A \arrow[l, "\varphi"']
\end{tikzcd}
$$

⌜ It follows from the axioms that c is an anti-homomorphism, i.e.

$$\begin{tikzcd}
A \otimes A \arrow[rr, "\varphi"] \arrow[d, "c \otimes c"] & & A \arrow[d, "c"] \\
A \otimes A \arrow[r, "tw"] & A \otimes A \arrow[r, "\varphi"] & A
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes A & & A \arrow[ll, "\psi"'] \\
A \otimes A \arrow[u, "c \otimes c"] & A \otimes A \arrow[l, "tw"'] & A \arrow[l, "\psi"'] \arrow[u, "c"']
\end{tikzcd}$$

But it is not true, in general, that $c \circ c = \mathrm{Id}$. ⌟

We say that A is <u>commutative</u> and <u>cocommutative</u>, respectively, if the following diagr. commute

$$\begin{tikzcd}
A \otimes A \arrow[rr, "tw"] \arrow[dr, "\varphi"'] & & A \otimes A \arrow[dl, "\varphi"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A & & A \otimes A \arrow[ll, "tw"'] \\
& A \arrow[ul, "\psi"] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

ex Let G be a (discrete) group, and let $k[G]$ be the free k-mod. with basis G. Let φ, ψ, η, ε, and c be the k-linear maps given by

$$\varphi(g \otimes g') = g g'$$

$$\psi(g) = g \otimes g$$

$$\eta(1) = 1$$

$$\varepsilon(g) = 1$$

$$c(g) = g^{-1}.$$

Then $k[G]$ is a cocomm. Hopf algebra.

ex Let $A = k[x]$ with the maps

$$\varphi(x^m \otimes x^n) = x^{m+n}$$

$$\psi(x^n) = \sum_{i+j=n} x^i \otimes x^j$$

$$\eta(1) = 1$$

$$\varepsilon(x^n) = \delta_{n,0}$$

$$c(x^n) = (-1)^{|x^n|} x^n.$$

Then $k[x]$ is a comm. + cocomm. Hopf alg.

ex Let G be a top'l group, or more generally, a group object in the cat. of top'l spaces and homotopy classes of maps, and let k be a field. Then $H_*(G) = H_*(G,k)$ is a cocomm. Hopf algebra with structure maps

$$H_*(G) \otimes H_*(G) \xrightarrow{\sim} H_*(G \times G) \xrightarrow{\mu_*} H_*(G)$$

$$H_*(G) \xrightarrow{\Delta_*} H_*(G \times G) \xleftarrow{\sim} H_*(G) \otimes H_*(G)$$

$$k = H_*(pt) \xrightarrow{e_*} H_*(G)$$

$$H_*(G) \xrightarrow{pr_*} H_*(pt) = k$$

$$H_*(G) \xrightarrow{inv_*} H_*(G).$$

We continue to write $H^*(X) = H^*(X, \mathbb{F}_2)$. Let $\mathcal{A}^* = \mathcal{A}^*_{\mathbb{F}_2}$ be the Steenrod algebra. In order to determine the multiplicative structure we consider the ring homomorphism

$$\mathcal{A}^* \longrightarrow \mathrm{End}(H^*(X))$$

for various X. We shall use the <u>Cartan formula</u>

$$Sq^i(xy) = \sum_{j \leq i} Sq^j(x) \cdot Sq^{i-j}(y).$$

More generally, if $I = (i_1, \ldots, i_r)$ is a sequence of non-neg. integers, then

$$Sq^I(xy) = \sum_{J \leq I} Sq^J(x) \cdot Sq^{I-J}(y).$$

Here $J = (j_1, \ldots, j_r) \leq I = (i_1, \ldots, i_r)$ means that $j_s \leq i_s$, for all $1 \leq s \leq r$, and

$$I - J = (i_1 - j_1, \ldots, i_r - j_r).$$

<u>Lemma</u> If $x \in H^1(X)$, then

$$Sq^i(x^k) = \binom{k}{i} u^{k+i} .$$

pf By induction on $k \geq 0$, $k=0$ o.k.

$$Sq^i(x^k) = Sq^i(x \cdot x^{k-1})$$

$$= Sq^0(x) \cdot Sq^i(x^{k-1}) + Sq^1(x) \cdot Sq^{i-1}(x^{k-1})$$

$$= x \cdot \binom{k-1}{i} x^{k-1+i} + x^2 \binom{k-1}{i-1} x^{k-1+i-1}$$

$$= \binom{k}{i} x^{k+i} .$$

//

Let $J_m := (2^{m-1}, \ldots, 2^1, 2^0)$, $m \geq 0$. Let J be a sequence of positive integers and let $x \in H^1(X)$. Then

$$Sq^J(x) = \begin{cases} x^{2^m} & , J = J_m \text{, some } m \geq 0, \\ 0 & , \text{else.} \end{cases}$$

Indeed, for all $0 < j < 2^k$,

$$\binom{2^k}{j} \equiv 0 \pmod 2 .$$

We recall that

$$H^*(\underbrace{\mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty}_{n}) = S_{\mathbb{F}_p}\{x_1, \ldots, x_n\}$$

where $x_i = pr_i^*(x)$ and $x \in H^1(\mathbb{R}P^1)$ is the unique generator.

Prop The following map is injective for $q \leq n$

$$\mathcal{A}^q \longrightarrow H^{n+q}(\underbrace{\mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty}_{n})$$

$$\theta \longmapsto \theta(x_1 \cdots x_n).$$

pf We recall that

$$\mathcal{A}^q = \mathbb{F}_2\{Sq^I \mid I \text{ adm.}, n(I) = q\}.$$

The proof is by induction on $n \geq 1$. We have $\mathcal{A}^0 = \mathbb{F}_2\{1\}$, $\mathcal{A}^1 = \mathbb{F}_2\{Sq^1\}$, and since $Sq^1(x) = x^2 \neq 0$, the case $n=1$ is o.k. So assume the statement for $n-1$, and let $q \leq n$ be fixed. Suppose

$$\sum_{n(I)=q} a_I \, Sq^I(w) = 0, \quad w = x_1 \cdots x_n.$$

We show that $a_I = 0$ by descending induction on $\ell(I)$. Here $\ell(I) = r$ if $I = (i_1, \dots, i_r)$. Suppose $a_I = 0$ if $\ell(I) > m$. such that

$$\sum_{\substack{\ell(I)=m \\ n(I)=q}} a_I Sq^I(w) + \sum_{\substack{\ell(I)<m \\ n(I)=q}} a_I Sq^I(w) = 0.$$

By the Cartan formula,

$$Sq^I(w) = Sq^I(x_1 . \underbrace{x_2 \cdots x_n}_{w'})$$

$$= \sum_{J \leq I} Sq^J(x_1) . Sq^{I-J}(w')$$

$$= \sum_{s \geq 0}^{\oplus} \sum_{\substack{J \leq I \\ n(J) = s}} Sq^J(x_1) . Sq^{I-J}(w').$$

Here $J \leq I$ is a seq. of non-negative integers. We consider the summand $s = 2^m$. By the lemma, $Sq^J(x_1) = 0$ unless $J = J_m$, and $Sq^{J_m}(x_1) = x_1^{2^m}$. We conclude

$$x_1^{2^m} \cdot \sum_{\substack{l(I)=m \\ n(I)=q}} a_I \, Sq^{I-J_m}(w') = 0 .$$

If we associate to I the seq. $I-J_m$, with possible zeros removed, we get a 1-1 correspondance between adm. ~~seq.~~ I, with $l(I) = m$ and $n(I) = q$, and adm. seq. I', with $l(I') \le m$ and $n(I') = q-(2^m-1)$. But $m \geq 1$ so $q-(2^m-1) \le n-1$. Done by induction. //

Consider the formal power series

$$P(t) = \sum_{i \ge 0} Sq^i \cdot t^i \in \mathcal{A}^*[[t]] .$$

<u>Thm</u> Let $u = 1+t$ and $\tau = ut$. Then

$$P(\tau) \cdot P(1) = P(u) P(t^2) .$$

pf By the previous result, it suffices

to show that for all $n \geq 1$,

$$P(\tau)P(1)(x_1 \cdots x_n) = P(u)\,P(t^2)(x_1 \cdots x_n)$$

as elements of $H^*(\underbrace{\mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty}_{n})[[t]]$. By the Cartan formula

$$P(\tau)P(1)(x_1 \ldots x_n)$$

$$= P(\tau)P(1)(x_1) \cdots P(\tau)P(1)(x_n)$$

and

$$P(u)P(t^2)(x_1 \cdots x_n)$$

$$= P(u)P(t^2)(x_1) \cdots P(u)P(t^2)(x_n),$$

so we can assume $n = 1$. In this case we simply calculate:

$$P(\tau)P(1)(x) = P(\tau)(x + x^2)$$

$$= x + (1+\tau)x^2 + \tau^2 x^4$$

$$P(u)P(t^2)(x) = P(u)(x + t^2x^2)$$

$$= x + (u + t^2)x^2 + u^2t^2x^4.$$

These are equal iff $u = 1+t$ and $\tau = ut$. //

$$\underline{\text{Cor}} \quad Sq_t^a Sq_t^b = \sum_{j=0}^{[\frac{a}{2}]} \binom{b-1-j}{a-2j} Sq_t^{a+b-j} Sq_t^j \quad , \quad a < 2b .$$

pf Set $\deg(t) = 0$. Then $Sq_t^a Sq_t^b$ is the coefficient of τ^a in $P(\tau)P(1)$ in degree $a+b$, i.e.

$$Sq_t^a Sq_t^b = \left[\operatorname{Res}_{\tau=0} P(\tau)P(1) \frac{d\tau}{\tau^{a+1}} \right]_{a+b}$$

$$= \left[\operatorname{Res}_{t=0} P(\tau)P(1) \frac{d\tau}{\tau^{a+1}} \right]_{a+b}$$

$$\overset{\text{thm.}}{=} \left[\operatorname{Res}_{t=0} P(u)P(t^2) \frac{dt}{(ut)^{a+1}} \right]_{a+b}$$

and this is the coeff. of t^a in

$$\left[\frac{P(u)P(t^2)}{u^{a+1}} \right]_{a+b} = \sum_{j \geq 0} Sq_t^{a+b-j} u^{b-j-1} Sq_t^j t^{2j}$$

Since $a < 2b$, only the summand with $j < b$ will contribute. Then $b-j-1 \geq 0$, so by the binomial formula

$$\sum_{0\le j<b} Sq^{a+b-j} w^{b-j-1} Sq^{j} t^{2j}$$

$$= \sum_{0\le j<b} Sq^{a+b-j} Sq^{j} \sum_{0\le i\le b-j-1} \binom{b-j-1}{i} t^{i+2j}$$

The coeff. of t^a is

$$\sum_{j=0}^{[a/2]} \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^{j} .$$

//

We have used the algebraic fact that the residue is indenpent of the param.

Let A be a ring, and consider the powerseries ring $A[[t]]$ and the ring of Laurent series $A((t))$. We have

$$\hat{\Omega}^1_{A((t))/A} = A((t)) \,.\, dt$$

and if $\omega = p(t)dt$ we define the residue $\mathrm{Res}_{t=0}(\omega)$ to be the coeff. of t^{-1} in $p(t)$. Suppose

$$\tau = t + a_2 t^2 + a_3 t^2 + \cdots$$

Then $A((t)) = A((\tau))$. We wish to show

<u>Lemma</u> $\mathrm{Res}_{\tau=0}(\omega) = \mathrm{Res}_{t=0}(\omega)$.

pf Suppose $p(t) = \sum_{n>>-\infty} a_n t^n$. Then we write $v(p) = n$ if $a_n \neq 0$ and $a_m = 0$, for $m < n$. We have:

(i) $\mathrm{Res}_{t=0}(\omega)$ is A-linear in ω.

(ii) $\mathrm{Res}_{t=0}(\omega) = 0$ if $\omega = f\,dt$, $f \in A[[t]]$.

(iii) $\mathrm{Res}_{t=0}(df) = 0$, $f \in A((t))$.

(iv) $\mathrm{Res}_{t=0}\left(\frac{dg}{g}\right) = v(g)$, $g \in A((t))^*$

The prop. (i) - (iii) are clear. To prove (iv) we write $g = t^n f$ with $f \in A[[t]]^*$ and $n = v(g)$. Then

$$\frac{dg}{g} = n\frac{dt}{t} + \frac{f'(t)}{f(t)}\,dt$$

so by (i) and (ii),

$$\mathrm{Res}_{t=0}\left(\frac{dg}{g}\right) = n\,\mathrm{Res}_{t=0}\left(\frac{dt}{t}\right) = n.$$

To prove the lemma, write

$$\omega = \sum_{n>0} \alpha_n \tau^{-n} d\tau + \omega_0$$

with $\omega_0 \in A[[\tau]]\cdot d\tau$. Then

$$\mathrm{Res}_{\tau=0}(\omega) = \alpha_1$$

$$\mathrm{Res}_{t=0}(\omega) = \sum_{n>0} \alpha_n \,\mathrm{Res}_{t=0}\left(\frac{d\tau}{\tau^n}\right).$$

By (iv), $\mathrm{Res}_{t=0}\left(\frac{d\tau}{\tau}\right) = 1$, so it suffices to show

$$\mathrm{Res}_{t=0}\left(\frac{d\tau}{\tau^n}\right) = 0, \qquad n \geq 2.$$

If A is a $\mathbb{Q}$-algebra, we let

$$g = -\frac{1}{n-1}\,\tau^{-(n-1)} \in A((\tau))^*$$

Then $dg = \tau^{-n} d\tau$ and $\mathrm{Res}_{t=0}(dg) = 0$ by (iii). In general, we have

$$\tau = t(1 + a_2 t + a_3 t^2 + \dots) \rightsquigarrow$$

$$\tau^{-n} = t^{-n}(1 - n a_2 t + \dots + b_i t^i + \dots)$$

where $b_i \in \mathbb{Z}[a_2, \dots, a_{i+1}]$ are certain pol. that depend only on n (and not on A). Multiplying by

$$d\tau = (1 + 2a_2 t + 3 a_3 t^2 + \dots)\, dt$$

we get that

$$\frac{d\tau}{\tau^n} = \frac{dt}{t^n} \cdot \sum_{i \geq 0} c_i t^i$$

where $c_i \in \mathbb{Z}[a_2, \dots, a_{i+1}]$ are certain (universal) polynomials. In particular,

$$c_{n-1} = \mathrm{Res}_{t=0}(\tau^{-n} d\tau).$$

The image of c_{n-1} by the incl.

$$\mathbb{Z}[a_2, \dots, a_n] \longrightarrow \mathbb{Q}[a_2, \dots, a_n]$$

is zero. Hence c_{n-1} is zero.

A __fiber bundle__ over X with fiber F is a map $E \xrightarrow{p} X$ which is locally trivial in the following sense. There exists a covering $\{U_i\}$ of X and homeomorphisms

$$\begin{array}{ccc} U_i \times F & \xrightarrow[\sim]{\phi_i} & p^{-1}(U_i) \\ \downarrow pr_1 & & \downarrow p \\ U_i & = & U_i \end{array}.$$

In particular, the fibers $F_x = p^{-1}(x)$ are all non-canonically homeomorphic to F.

__Prop__ A fiber bundle is a Serre fibration.

__pf__ It suffices to show that in the following diagram, a lift exists.

$$\begin{array}{ccc} \Delta^0 & \longrightarrow & E \\ \int & \nearrow & \downarrow p \\ \Delta^n & \xrightarrow{f} & X \end{array}.$$

Pick a cov. $\{U_i\}$ of X and homeo.

$\{U_i \times F \xrightarrow{\phi_i} p^{-1}(U_i)\}$. Then $\{f^{-1}(U_i)\}$ cover Δ^n. We can subdivide Δ^n such that each simplex in the subdivision lies in one $f^{-1}(U_i)$. The lifting is now constructed by easy induction over the simplices. //

The maps

$$\begin{array}{ccccc} (V_i \cap V_j) \times F & \xrightarrow[\sim]{\phi_i} & p^{-1}(V_i \cap V_j) & \xleftarrow[\sim]{\phi_j} & (V_i \cap V_j) \times F \\ \downarrow pr_1 & & \downarrow p & & \downarrow pr_1 \\ V_i \cap V_j & = & V_i \cap V_j & = & V_i \cap V_j \end{array}$$

give rise to a homeomorphism

$$F \xrightarrow[\sim]{\phi_{j,x}^{-1}\,\phi_{i,x}} F$$

for each $x \in V_i \cap V_j$. With the current definition, this can be <u>any</u> homeo. of F. We want to restrict the homeo. that we allow. If, for example, F is a vector space, we want to limit the allowable homeomorphisms to be the linear isomorphisms. (This will be called a vecter bundle.)

Let G be a topological group that acts continously on F from the left,

$$G \times F \xrightarrow{\mu} F.$$

We assume that the action is <u>effective</u>, i.e. that if $g \cdot x = x$, for all $x \in F$, then $g = e$, or equivalently, that the adjoint map

$$G \xrightarrow{\tilde{\mu}} \mathrm{Aut}(F)$$

is injective. A <u>coordinate bundle</u> over X with fiber F and group G is a map

$$E \xrightarrow{p} X$$

together with coordinate functions

$$\begin{array}{ccc} V_i \times F & \underset{\sim}{\xrightarrow{\phi_i}} & p^{-1}(V_i) \\ \downarrow{\scriptstyle pr} & & \downarrow{\scriptstyle p} \\ V_i & = & V_i \end{array}$$

for a covering $\{V_i\}$ of X and (cts.)

transition functions

$$V_i \cap V_j \xrightarrow{g_{ji}} G$$

such that for all $x \in V_i \cap V_j$ and $y \in F$,

$$\phi_{j,x}^{-1} \phi_{i,x}(y) = g_{ji}(x) \cdot y .$$

We say that the coord. bdl. $\mathcal{E}'$ is a _refinement_ of the coord. bdl. $\mathcal{E}$, if $\mathcal{E}'$ and $\mathcal{E}$ have the same total space, base space, fiber, and group, and if the covering $\{V_i'\}$ is a refinement of the covering $\{V_i\}$. Two coord. bdls. are _equivalent_ if they have a common refinement. A _fiber bundle_ with fiber F and group G is an equivalence class of coord. bdls. with fiber F and group G.

Let $\mathcal{E}$ and $\mathcal{E}'$ be two coordinate bdls. with fiber F and group G, and suppose that we have a pull-back square

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[d, "p"] & E' \arrow[d, "p'"] \\ X \arrow[r, "\bar{h}"] & X' \end{tikzcd}$$

Then the structure of coord. bdl. on E' induces one on E. We say that $(h, \bar{h})$ is a <u>map</u> of coord. bdls. if this induced str. is equivalent to the given str. $\mathcal{E}$.

<u>Exercise</u> Show that the pull-back sq. $(h, \bar{h})$ is a map of coord. bdls. if and only if there exists cts. maps

$$V_i \cap \bar{h}^{-1}(V'_{i'}) \xrightarrow{\bar{g}_{i'i}} G$$

such that for all $x \in V_i \cap \bar{h}^{-1}(V'_{i'})$, and all $y \in F$,

$$\phi'^{-1}_{i',x'} \, h_x \, \phi_{i,x}(y) = \bar{g}_{i'i}(x) \cdot y \qquad //$$

<u>Exercise</u> Let $\mathcal{E}$ and $\mathcal{E}'$ be two coord. bdls. with the same fiber F, group G,

and base X, and suppose

$$\begin{array}{ccc} E & \xrightarrow{h} & E' \\ \downarrow p & \text{p.b.} & \downarrow p' \\ X & = & X \end{array}$$

is a map of coord. bdls. Show that this map is an isomorphism, i.e. that the inverse h^{-1} is cts. and a map of coord. bdls. //

If $\mathcal{E}$ is a coord. bdl. then the transition functions

$$V_i \cap V_j \xrightarrow{g_{ji}} G$$

satisfy the <u>cocycle</u> <u>condition</u>: for all $x \in V_i \cap V_j \cap V_k$,

$$g_{kj}(x) \cdot g_{ji}(x) = g_{ki}(x).$$

We show the following converse.

<u>Prop</u> Let G be a topological group that acts continuously and effectively from the left on the space F. Let $\{V_i\}$

be a covering of X, and let

$$V_i \cap V_j \xrightarrow{g_{ji}} G$$

be cts. maps that satisfy the cocycle condition. Then there exists a coord. bdl. over X with fiber F, group G, and transition functions g_{ji}. Any two such coord. bdls. are isomorphic.

pf Let E be the quotient space

$$E = \Big(\coprod_{i \in I} V_i \times F \Big) / \sim$$

where

$$(x,y)_i \sim (x',y')_j$$

if $x = x' \in V_i \cap V_j$ and $g_{ji}(x) \cdot y = y'$. The proj. $E \xrightarrow{p} X$ is given by $p((x,y)_i) = x$. We define a coord. fct.

$$\begin{aligned} V_i \times F &\xrightarrow{\phi_i} p^{-1}(V_i) \\ (x,y) &\longmapsto [(x,y)_i] \end{aligned}$$

It is onto: ~~If~~ $[(x,y)_j] \in p^{-1}(V_i)$, then $x \in V_i \cap V_j$. So $(x,y)_j \sim (x, g_{ij}(x)y)_i$ or

$$[(x,y)_j] = \phi_i(x, g_{ij}(x)y).$$

It is 1-1: If $\phi_i(x,y) = \phi_i(x',y')$ then $x = x'$ and $y = g_{ii}(x)y = y'$. So ϕ_i is a cts. bijection. We must show that ϕ_i^{-1} is cts., or equivalently, that ϕ_i is open. Let $W \subset V_i \times F$ be open. Since

$$\coprod_{j \in I} (V_i \cap V_j) \times F \xrightarrow{pr} p^{-1}(V_i)$$

is a quotient map, it suffices to show that $pr^{-1}(\phi_i(W))$ is open. But we have a comm. diagr.

$$\begin{tikzcd}
\coprod_{j \in I} (V_i \cap V_j) \times F \arrow[rr, "pr"] \arrow[dr, "r"'] & & p^{-1}(V_i) \\
& V_i \times F \arrow[ur, "\phi_i"'] &
\end{tikzcd}$$

where

$$r((x,y)_j) = (x, g_{ij}(x) \cdot y).$$

Now r is cts., and $pr^{-1}(\phi_i(W)) = r^{-1}(W)$. //

<u>Lemma</u> Two coord. bdls. $\mathcal{E}$ and $\mathcal{E}'$ with the same base X, fiber F, and group G, are isomorphic if and only if there exists cts. maps

$$V_i \cap V_j' \xrightarrow{\bar g_{ji}} G \quad , i \in I, \ j \in I',$$

s.t.

$$\bar g_{ki}(x) = \bar g_{kj}(x)\, g_{ji}(x) \quad , \ x \in V_i \cap V_j \cap V_k'$$

$$\bar g_{ki}(x) = g'_{kj}(x)\, \bar g_{ji}(x) \quad , \ x \in V_i \cap V_j' \cap V_k' .$$

<u>pf</u> Suppose that

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[d, "p"] \arrow[dr, phantom, "\text{p.b.}"] & E' \arrow[d, "p'"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

is an isomorphism $\mathcal{E} \xrightarrow{h} \mathcal{E}'$. Then

$$F \xrightarrow[\sim]{\phi'^{-1}_{j,x}\, h_x\, \phi_{i,x}} F$$

is mult. by a uniquely determined elem. $\bar g_{ji}(x) \in G$, and the map so defined

$$V_i \cap V_j' \xrightarrow{\bar{g}_{ji}} G$$

is cts. Conversely, these maps determine a homeo. h that is an iso. //

The prop. and lemma shows that the set

$$\left\{ \begin{array}{l} \text{iso. classes of fiber bdls.} \\ \text{over } X \text{ with fiber } F \text{ and} \\ \text{group } G \end{array} \right\}$$

only depends on X and G up to canonical bijection.

A <u>principal</u> G-<u>bundle</u> is a fiber bdl. with group G and fiber G, with G acting on itself by left multiplication.

<u>Lemma</u> Let $E \xrightarrow{p} X$ be a principal G-bdl. Then E has a free right G-action, and p induces a homeo.

$$E/G \xrightarrow[\sim]{p} X.$$

pf In local coordinates the right G-action is given by

$$\begin{tikzcd} V_i \times G \times G \arrow[r, "\phi_i \times id", "\sim"'] \arrow[d, "id \times \mu"] & p^{-1}(V_i) \times G \arrow[d, "\mu_i"] \\ V_i \times G \arrow[r, "\phi_i", "\sim"'] & p^{-1}(V_i) \end{tikzcd}.$$

Since the transition functions in the (coord.) bdl. are given by left mult. by elem. of G, the μ_i, $i \in I$, are compatible. The action is free and $E/G \xrightarrow{\sim} X$. //

Prop Let G act freely and continuously on E from the right, and let

$$E \xrightarrow{p} E/G =: X$$

be the canonical projection. Suppose that there exists a covering $\{V_i\}$ of X and G-equivariant homeo.

$$\begin{tikzcd} V_i \times G \arrow[r, "\phi_i", "\sim"'] \arrow[d] & p^{-1}(V_i) \arrow[d, "p"] \\ V_i \arrow[r, equal] & V_i \end{tikzcd}$$

Then $E \xrightarrow{p} X$ is the bundle proj. of a uniquely determined principal G-bdl $\mathcal{E}$.

pf Note that a homeomorphism

$$G \underset{\sim}{\xrightarrow{f}} G$$

that is equivariant for right mult. by G on itself is given by left mult. by the elem. $f(1)$. Indeed,

$$f(g'g) = f(g')g \underset{g'=1}{\Longrightarrow}$$

$$f(g) = f(1)g.$$

The claim follows. //

We saw last time that there is a natural 1-1 correspondance between fiber bdls. with fiber F and group G and principal G-bdls. We can make this explicit as follows. Let $P \xrightarrow{p} X$ be a principal G-bdl. and define

$$E = P \times_G F := (P \times F)/G$$

$$(a,y)\cdot g = (ag, g^{-1}y)$$

Then

$$\begin{aligned} E &\longrightarrow X \\ [a,y] &\longmapsto p(x) \end{aligned}$$

is the projection in a unique fiber bdl. $\mathcal{E}$ over X with fiber F and group G. Conversely, let $\mathcal{E}$ be a fiber bdl. over X with fiber F and group G. Let $P(\mathcal{E})$ be the set of bdl. maps

$$\begin{tikzcd} F \arrow[r, "h"] \arrow[d] & E \arrow[d, "p"] \\ * \arrow[r, "\bar{h}"] & X \end{tikzcd}$$

The group G acts freely from the right on $P(\mathcal{E})$ as follows. If $(h, \bar{h})$ is a bdl. map and $g \in G$, then $(h,\bar{h})\cdot g$ is the bdl. map

$$\begin{array}{ccccc} F & \xrightarrow{g\cdot} & F & \xrightarrow{h} & E \\ \downarrow & & \downarrow & & \downarrow p \\ * & = & * & \xrightarrow{\bar{h}} & X . \end{array}$$

Finally, there is a unique topology on $P(\mathcal{E})$ s.t. the map

$$P(\mathcal{E}) \xrightarrow{q} X$$
$$(h, \bar{h}) \longmapsto \bar{h}(*)$$

is the projection of a principal G-bdl. over X. This is the <u>assoc</u>. <u>princ</u>. G-<u>bdl</u>. of the fiber bdl. $\mathcal{E}$. The constructions above are functors and define an equivalence of categories

$$\left\{ \begin{array}{l} \text{principal} \\ G\text{-bdls.} \end{array} \right\} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \left\{ \begin{array}{l} \text{fiber bdls. with} \\ \text{fiber } F \text{ and grp. } G \end{array} \right\}$$

Let $\mathcal{E}$ be a fiber bdl. over X with fiber F and group G. We write $\mathcal{E} \times [0,1]$ for the bdl. over $X \times [0,1]$ induced by the proj. $X \times [0,1] \to X$:

$$\begin{array}{ccc} E \times [0,1] & \longrightarrow & E \\ \big\downarrow p \times id & \text{p.b.} & \big\downarrow p \\ X \times [0,1] & \longrightarrow & X \end{array}$$

A bdl. homotopy is a bdl. map.

$$\mathcal{E} \times [0,1] \longrightarrow \mathcal{E}'.$$

Thm Let $\mathcal{E}$ and $\mathcal{E}'$ be fiber bdls. with the same fiber and group. Let $(h_0, \bar{h}_0)$ be a bdl. map from $\mathcal{E}$ to $\mathcal{E}'$, and let $X \times [0,1] \xrightarrow{\bar{h}} X'$ be a htpy. of $\bar{h}_0$. Suppose X is cpt. and Hausdorff. Then $\bar{h}$ can be lifted to a bdl. htpy.

$$\mathcal{E} \times [0,1] \xrightarrow{(h,\bar{h})} \mathcal{E}'.$$

Pf Since $X \times [0,1]$ is cpt., the covering $\{\bar{h}^{-1}(V_i)\}$ has a refinement of the form $\{U_\lambda \times I_\nu\}$ where $\{U_\lambda\}$ is an open cover of X and $I_1, \ldots, I_r$ a finite seq. of open intervals s.t. $I_\nu \cap I_\mu = \emptyset$ unless $\mu = \nu - 1, \nu, \nu+1$. Choose

$$0 = t_0 \leq t_1 \leq \cdots \leq t_{r-1} \leq t_r = 1$$

s.t. $t_\nu \in I_\nu \cap I_{\nu+1}$, $1 \leq r \leq \nu$. We assume, inductively, that h has been defined on $E \times [0, t_\nu]$. We must define h on $E \times [t_\nu, t_{\nu+1}]$. For $x \in X$, choose $x \in W, W' \subset X$ s.t. $\bar{W} \subset W'$ and $\bar{W}' \subset U_\lambda$, some λ. (Can do this since X is normal.) Since X is cpt., we can find a find a finite number of such pairs

$$(W_\alpha, W'_\alpha), \quad 1 \leq \alpha \leq s,$$

s.t. $\bigcup_{1 \leq \alpha \leq s} W_\alpha = X$. By Urysohn's lemma, there are cts. maps

$$X \xrightarrow{u_\alpha} [t_\nu, t_{\nu+1}], \quad 1 \leq \alpha \leq s,$$

s.t. $u_\alpha(\overline{W}_\alpha) = t_{\nu+1}$ and $u_\alpha(X \setminus W'_\alpha) = t_\nu$. Def.

$$X \xrightarrow{\tau_\alpha} [t_\nu, t_{\nu+1}], \quad 0 \leq \alpha \leq s,$$

$$\tau_\alpha(x) = \begin{cases} t_\nu & , \alpha = 0 \\ \max\{u_1(x), \ldots, u_\alpha(x)\} & , 1 \leq \alpha \leq s, \end{cases}$$

Then

$$t_\nu = \tau_0(x) \leq \tau_1(x) \leq \cdots \leq \tau_s(x) = t_{\nu+1}.$$

Define

$$X_\alpha = \{(x,t) \in X \times [0,1] \mid t_\nu \leq t \leq \tau_\alpha(x)\}$$

$$E_\alpha = (p \times id)^{-1}(X_\alpha).$$

Then

$$E \times \{t_\nu\} = E_0 \subset E_1 \subset \cdots \subset E_s = E \times [t_\nu, t_{\nu+1}].$$

Assume, inductively, that h has been defined on $E_{\alpha-1}$; we extend h to E_α.

$$X_\alpha \setminus X_{\alpha-1} \subset W'_\alpha \times [t_\nu, t_{\nu+1}] \subset U_\lambda \times [t_\nu, t_{\nu+1}]$$

$$\subset \bar{h}^{-1}(V'_j).$$

Let

$$p'_j : p'^{-1}(V'_j) \xleftarrow[\sim]{\phi'_j} V'_j \times F' \xrightarrow{pr_2} F'$$

and define, for $(e,t) \in E_\alpha \setminus E_{\alpha-1}$,

$$h(e,t) = \phi'_j\left(\bar{h}(x,t), p'_j h(e, \tau_{\alpha-1}(x))\right),$$

$$x = p(e).$$

One checks that this extends h to a bdl. map on E_α. //

Cor Every bdl. over $X \times [0,1]$ is isomorphic to a bdl. of the form $\mathcal{E} \times [0,1]$.

pf Let $\mathcal{E}'$ be a bdl. over $X \times [0,1]$, and

$$\begin{tikzcd} \mathcal{E} := \bar{i}_0^* \mathcal{E}' \arrow[r, "i_0"] \arrow[d, "p"] & \mathcal{E}' \arrow[d, "p'"] \\ X \arrow[r, "\bar{i}_0"] & X \times [0,1] \end{tikzcd}$$

$$x \longmapsto (x, 0)$$

Since the identity map on $X \times [0,1]$ is a htpy. of $\bar{i}_0$ (to $\bar{i}_1$), the theorem given a bdl. map

$$\begin{tikzcd} \mathcal{E} \times [0,1] \arrow[r, "h"] \arrow[d] & \mathcal{E}' \arrow[d] \\ X \times [0,1] \arrow[r, equal] & X \times [0,1] \end{tikzcd}$$

This is an iso. (by exercise). //

<u>Cor</u> Let $\mathcal{E}'$ be a bdl. over X', and let $f, g : X \to X'$ be homotopic maps. Then the induced bdls. $f^*\mathcal{E}'$ and $g^*\mathcal{E}'$ over X are isomorphic.

<u>pf</u> Let $X \times [0,1] \xrightarrow{h} X'$ be a htpy from f to g. By the previous cor.,

$$(f^*\mathcal{E}') \times [0,1] \xrightarrow{\sim} h^*\mathcal{E}'$$

as bdls. over $X \times [0,1]$. If we pull back along $i_1 : X \to X \times [0,1]$, we get

$$f^*\mathcal{E}' \xrightarrow{\sim} g^*\mathcal{E}'$$

as desired. //

A <u>universal</u> G-<u>bundle</u> is a principal G-bdl.

$$EG \xrightarrow{p} BG$$

s.t. for every fin. rel. CW-cx. (X, A), every principal G-bdl. $E \to X$, and every bdl. map $f : E|_A \to EG$, there exists a bdl. map $E \to EG$ that extends f.

<u>Prop</u> A principal G-bdl. $E \to X$ is universal if and only if $\pi_i(E) = 0$, for all $i \geq 0$.

<u>pf</u> If Y is a space and Z a right G-space, then

$$\mathrm{Map}_G(Y \times G, Z) \xrightarrow{\sim} \mathrm{Map}(Y, Z).$$
$$f \longmapsto f(-, e)$$

Suppose $\pi_i(E) = 0$, for all $i \geq 0$. We a cover $\{V_i\}$ of X and G-equivariant coord. fcts.

$$V_i \times G \underset{\sim}{\xrightarrow{\phi_i}} p^{-1}(V_i).$$

We can assume that each cell of (X,A) is contained in some V_i. It follows that $(E, p^{-1}(A))$ is a free G-CW-cx:

$$\begin{tikzcd} \coprod S^{n-1}\times G \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "p.o."] & sk_{n-1}E \arrow[d, hook] \\ \coprod D^n \times G \arrow[r] & sk_n E \end{tikzcd}$$

This is a p.o. diagr. of right G-spaces. We now extend a G-map $sk_{-1}E = p^{-1}(A) \to EG$ to E by ind. over the skeleta: every G-map

$$\coprod S^{n-1}\times G \longrightarrow EG$$

can be extended over $\coprod D^n \times G$. //

Thm Let X be a finite CW-cx., and let $EG \to BG$ be a universal G-bdl. Then there is a natural bijection

$$[X, BG] \longrightarrow \left\{ \begin{array}{c} \text{iso. classes of} \\ \text{principal } G\text{-bdls} \\ \text{over } X \end{array} \right\}$$

$$[f] \longmapsto [f^*EG \to X].$$

pf The map is well-defined by the last cor. To see that it is onto, let $E \to X$ be a principal G-bdl. over X. Then there exists a bdl. map

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & EG \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & BG \end{array}.$$

The induced bdl. map $E \to f^*EG$ of bdls. over X is (automatically) an iso. So the map is onto.

To show that it is 1-1, we must show that

$$\begin{array}{ccc} f^*EG & \xrightarrow{h} & g^*EG \\ \downarrow & & \downarrow \\ X & = & X \end{array} \Rightarrow f \simeq g : X \to BG.$$

Let $E = f^*EG$ and consider

$$\begin{array}{ccccc} E\times\{0\} \amalg E\times\{1\} & \xrightarrow{id \amalg h} & f^*EG \amalg g^*EG & \longrightarrow & EG \\ \downarrow & & \downarrow & & \downarrow \\ X\times\{0\} \amalg X\times\{1\} & = & X \amalg X & \xrightarrow{f \amalg g} & BG \end{array}.$$

Since $EG \to BG$ is universal, it extends to a bdl. htpy

$$\begin{array}{ccc} E \times [0,1] & \longrightarrow & EG \\ \downarrow & & \downarrow \\ X \times [0,1] & \longrightarrow & BG . \end{array}$$

In particular, f and g are htpic. //

We now construct a universal G-bdl. for every topological group G s.t. the inclusion $\{e\} \hookrightarrow G$ of the identity elem. has the homotopy extension property. In short EG is defined to be the configuration space of particles on the interval $[0,1]$ with labels (or charges) in G s.t. labels multiply if two particles collide, particles with label $e \in G$ can be omitted / added, and particles are annihilated at $0 \in [0,1]$. We define BG analogously except that in BG, particles are annilated also at $1 \in [0,1]$. In more detail, let

$$\Delta^n = \{ x \in \mathbb{R}^n \mid 0 \leq x_1 \leq \cdots \leq x_n \leq 1 \} ;$$

then

$$EG = \Big(\coprod_{n \geq 0} \Delta^n \times G^n \Big) / \sim$$

where the equivalence rel. identifies

$$(x_1,\dots,x_i,x_i,\dots,x_n;\, g_1,\dots,g_i,g_i',\dots,g_n)$$

$$\sim (x_1,\dots,x_i,\dots,x_n;\, g_1,\dots,g_i g_i',\dots,g_n)$$

$$(x_1,\dots,x_i,\dots,x_n;\, g_1,\dots,e,\dots,g_n)$$

$$\sim (x_1,\dots,\hat{x}_i,\dots,x_n;\, g_1,\dots,\hat{e},\dots,g_n)$$

$$(0,x_2,\dots,x_n;\, g_1,g_2,\dots,g_n)$$

$$\sim (x_2,\dots,x_n;\, g_2,\dots,g_n).$$

Similarly, we let

$$BG = \Big(\coprod_{n\geq 0} \Delta^n \times G^n\Big)\Big/ \approx \,,$$

where in addition to the rel. above,

$$(x_1,\dots,x_{n-1},1;\, g_1,\dots,g_{n-1},g_n)$$

$$\approx (x_1,\dots,x_{n-1};\, g_1,\dots,g_{n-1}),$$

and define

$$EG \xrightarrow{f} BG$$

to be the canonical projection. (We give EG and BG the obvious topology.) The group G acts freely on EG by

$$[x_1, \ldots, x_n ; g_1, \ldots, g_n] \cdot g$$

$$= [x_1, \ldots, x_n, 1 ; g_1, \ldots, g_n, g],$$

and f induces a homeomorphism

$$EG/G \xrightarrow[\sim]{f} BG.$$

In fact, EG is a topological group with the multiplication given by

$$[x_1, \ldots, x_n ; g_1, \ldots, g_n] \cdot [x_1, \ldots, x_n, \bar{g}_1, \ldots, \bar{g}_n]$$

$$= [x_1, \ldots, x_n ; g_1 g_2 \cdots g_n \bar{g}_1 g_n^{-1} \cdots g_2^{-1}, \ldots, g_n \bar{g}_n].$$

Note that the x-coordinates are the same. We can alway arrange this by adding particles with label $e \in G$. We can then view G as the closed subgroup of EG consisting of the configurations $[1; g]$, $g \in G$.

The space EG is contractible with a canonical null-homotopy given by

$$EG \times [0,1] \xrightarrow{h} EG$$

$$([x_1,\dots,x_n;g_1,\dots,g_n],t) \mapsto [x_1t,\dots,x_nt;g_1,\dots,g_n].$$

We note that h is well-defined, and also that this formula does not define a null-htpy. of BG, since the extra rel. is not preserved.

Thm Suppose $\{e\} \hookrightarrow G$ has the homotopy extension property. Then

$$EG \xrightarrow{f} BG$$

is a principal G-bdle, and hence a universal G-bdle.

pf See J. P. May: "Classifying spaces and fibrations", Mem. Amer. Math. Soc. vol. 1, no. 155 (1975), Thm. 8.2. But note that to give a local trivialisection

$$\begin{tikzcd} V \times G \arrow[r, "\phi"] \arrow[d] & f^{-1}(V) \arrow[d, "f"] \\ V \arrow[r, equal] & V \end{tikzcd}$$

is equivalent to giving a <u>section</u> of f <u>over</u> V, i.e. a map σ,

$$\begin{tikzcd} f^{-1}(V) \arrow[d, shift right, "f"'] \\ V \arrow[u, bend right, "\sigma"'] \end{tikzcd} \qquad f \circ \sigma = \mathrm{id}_V$$

Indeed, given ϕ, we let $\sigma(v) = \phi(v, e)$, and given σ, we let $\phi(v, g) = \sigma(v) g$. //

We note that if a CW-structure on G gives rise to one on BG. Also, if $G \xrightarrow{\alpha} G'$ is a group homomorphism and a weak equivalence, then

$$BG \underset{\sim}{\xrightarrow{\alpha_*}} BG'$$

again is a w.e.

let $c \in H^*(BG)$ be a coh. class. Then, if E is a principal G-bdl. over X, we get an induced coh. class

$$\begin{array}{ccc} H^*(BG) & \xrightarrow{f^*} & H^*(X) \\ c & \longmapsto & c(E) \end{array},$$

where $X \xrightarrow{f} BG$ is s.t. $E \xrightarrow{\sim} f^*EG$. Clearly, for every map $X' \xrightarrow{g} X$,

$$(*) \qquad c(g^*E) = g^*(c(E))$$

in $H^*(X')$. We say that a coh. cl. $c(E) \in H^*(X)$ associated with a principal G-bdl. E over X is a <u>characteristic</u> <u>class</u> if $(*)$ holds. Then, by the Yoneda lemma, every char. class arises as

$$c(E) = f^*(c),$$

for some (universal) class $c \in H^*(BG)$. We consider $GL_n(\mathbb{C})$-bundles.

<u>Lemma</u> There is a fiber sequence

$$G/H \longrightarrow BH \longrightarrow BG.$$

Pf We can use EG as our model for EH. Then we get a fiber bdl.

$$\begin{array}{ccccc} G/H & \longrightarrow & EG/H & \longrightarrow & EG/G \\ & & \| & & \| \\ & & BH & \longrightarrow & BG \end{array} \qquad - \qquad ''$$

We use the lemma to calc. the coh. of $BGL_n(\mathbb{C})$. We include

$$GL_{n-1}(\mathbb{C}) \hookrightarrow GL_n(\mathbb{C})$$

$$A \longmapsto \begin{pmatrix} A & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ 0 \cdots 0 & 1 \end{pmatrix}$$

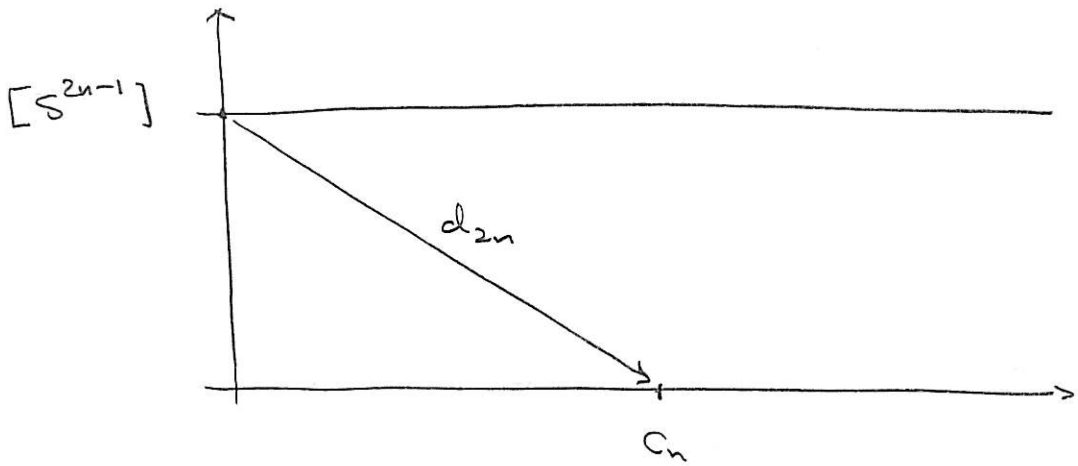
Reading off the last collumn gives rise to a homeomorphism

$$GL_n(\mathbb{C}) / GL_{n-1}(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}^n \setminus \{0\}$$

so we have a fiber sequence

$$\begin{array}{l} \mathbb{C}^n \setminus \{0\} \longrightarrow BGL_{n-1}(\mathbb{C}) \longrightarrow BGL_n(\mathbb{C}). \\ \quad \wr\| \\ S^{2n-1} \end{array}$$

The Serre sp. seq. takes the form



We show, by induction, that

$$H^*(BGL_n(\mathbb{C})) \approx S_{\mathbb{Z}}\{c_1, \ldots, c_n\}.$$

Assume the statement for $n-1$. Since the coh. is conc. in even degrees ($\deg c_i = 2i$), the diff. indicated above must be an iso. of $E_{2n}^{0,2n-1} = E_2^{0,2n-1}$ onto the subgroup $\mathbb{Z} \cdot c_n \subset E_{2n}^{2n,0} = E_2^{2n,0}$. We show that the coh. of $GL_n(\mathbb{C})$ is conc. in even degrees. If not, let x be a non-zero class of odd degree s, and suppose s is minimal with this property. But then there must be a non-zero diff

$$E_{2n}^{s-2n,\,2n-1} \xrightarrow{d_{2n}} E_{2n}^{s,\,0}$$

or else x survive to $E_\infty^{s,0}$. But

$$E_{2n}^{s-2n,\,2n-1} \xleftarrow{\sim} E_{2n}^{s-2n,\,0} \otimes E_{2n}^{0,\,2n-1},$$

and $E_{2n}^{s-2n,\,0} = 0$ since $s-2n < s$ and odd. Thus $H^*(BGL_n(\mathbb{C}))$ is conc. in even degrees. It follows that

$$E_{2n}^{s,\,2n-1} \xrightarrow[\sim]{d_{2n}} E_{2n}^{s+2n,\,0},$$

for all $s \geq 0$. It follows that

$$H^*(BGL_n(\mathbb{C})) = S_{\mathbb{Z}}\{c_1, \ldots, c_n\},$$

where for $1 \leq i < n$, c_i is the unique class which maps to the elem. with same name under

$$H^*(BGL_n(\mathbb{C})) \xrightarrow{\iota^*} H^*(BGL_{n-1}(\mathbb{C})).$$

The classes c_i, $1 \leq i \leq n$, are (up to a sign) the universal Chern classes. We fix the sign next time.

A real vector bdl. of rk. n is a fiber bdl. with fiber $\mathbb{R}^n$ and group $GL_n(\mathbb{R})$. The coordinate functions give each fiber in the bdl. the structure of a real vector space of rk. n, and two coord. bdls. are equivalent iff and only if they give rise to the <u>same</u> vector space structure on the fibers. So an equivalent definition of a real vector bdl of rk. n is as follows:

A cts. map

$$E \xrightarrow{p} X,$$

and for all $x \in X$, a structure of real vector space of rk. n on $E_x = p^{-1}(x)$ s.t. the following local triviality condition holds. There exists an open covering $\{V_i\}$ of X and homeomorphisms

$$\begin{array}{ccc} V_i \times \mathbb{R}^n & \xrightarrow[\sim]{\phi_i} & p^{-1}(V_i) \\ \downarrow pr_1 & & \downarrow p \\ V_i & = & V_i \end{array}$$

s.t. for all $x \in V_i$, the map

$$\mathbb{R}^n \xrightarrow[\sim]{\phi_{i,x}} E_x$$

is <u>linear</u>. We define complex v.b. similarly.

Let E and E' be two real vector bdls. over X of rk. n and n'. We construct a v.b. $E \oplus E'$ of rk. $n+n'$ s.t.

$$(E \oplus E')_x = E_x \oplus E'_x ,$$

for all $x \in X$. Here is one way to do this: Write

$$E = P \times_{GL_n(\mathbb{R})} \mathbb{R}^n , \quad E' = P' \times_{GL_{n'}(\mathbb{R})} \mathbb{R}^{n'}$$

Then

$$E \oplus E' = (P \underset{X}{\times} P') \times_{GL_n(\mathbb{R}) \times GL_{n'}(\mathbb{R})} (\mathbb{R}^n \times \mathbb{R}^{n'})$$

$$= \left((P \underset{X}{\times} P') \times_{GL_n(\mathbb{R}) \times GL_{n'}(\mathbb{R})} GL_{n+n'}(\mathbb{R}) \right) \times_{GL_{n+n'}(\mathbb{R})} \mathbb{R}^{n+n'}$$

In the same way, other constructions on vector spaces generalizes to vector bundles, e.g.

$$E \otimes E', \ \mathrm{Hom}(E, E'), \ E^*, \ \Lambda^i E, \ \ldots$$

Let E be a v.b. over X. By a <u>subbundle</u> of E, we mean v.b. E' over X s.t. $E' \subset E$ and s.t. for all $x \in X$, $E'_x \subset E_x$ is a sub-vector space.

<u>Lemma</u> Let $E' \subset E$ be a sub-bundle. Then there exists a covering $\{V_i\}$ of X and coordinate functions

$$\begin{tikzcd} V_i \times \mathbb{R}^n \arrow[r, "\phi_i"] \arrow[d, "pr_1"] & p^{-1}(V_i) \arrow[d, "p"] \\ V_i \arrow[r, equal] & V_i \end{tikzcd}$$

whose restriction to $V_i \times \mathbb{R}^{n'} \subset V_i \times \mathbb{R}^n$ give rise to coordinate functions on E'.

Pf Let $x_0 \in X$ and choose coordinate functions on E and E' around x_0. We can assume that the assoc. open nbhds. of x are equal. (If not, use intersection.)

$$\begin{array}{ccccc} V\times\mathbb{R}^{n'} & \xrightarrow[\sim]{\phi'} & p'^{-1}(V) & = & E'\cap p^{-1}(V) \\ \downarrow pr_1 & & \downarrow p' & & \downarrow p \\ V & = & V & = & V \end{array}$$

$$\begin{array}{ccc} V\times\mathbb{R}^{n} & \xrightarrow[\sim]{\phi} & p^{-1}(V) \\ \downarrow pr_1 & & \downarrow p \\ V & = & V \end{array}$$

We can assume that

$$\mathbb{R}^{n'} \xrightarrow{\phi_{x_0}^{-1}\,\phi'_{x_0}} \mathbb{R}^n$$

is equal to the standard inclusion $(t_1,\dots,t_{n'}) \mapsto (t_1,\dots,t_{n'},0,\dots,0)$. (If not, we compose ϕ_x, for all $x\in V$, by a fixed $\alpha\in GL_n(\mathbb{R})$.) Consider the map

$$V \xrightarrow{\alpha} M_n(\mathbb{R})$$

$$x \longmapsto \left(\begin{array}{c|c} \phi_x^{-1}\phi'_x & 0 \\ \hline & I_{n-n'} \end{array}\right)$$

Let $V' \subset V$ be the inverse image of $\mathbb{R} \setminus \{0\}$ by the composite

$$V \xrightarrow{\alpha} M_n(\mathbb{R}) \xrightarrow{\det} \mathbb{R}.$$

Then $x_0 \in V' \subset X$ is open. We replace V by V' and ϕ by the composite

$$\begin{array}{ccccc} V' \times \mathbb{R}^n & \xrightarrow[\sim]{\tilde{\alpha}} & V' \times \mathbb{R}^n & \xrightarrow[\sim]{\phi} & p^{-1}(V') \\ \downarrow pr_1 & & \downarrow pr_1 & & \downarrow p \\ V' & = & V' & = & V' \end{array},$$

where $\tilde{\alpha}(x, v) = (x, \alpha(x) \cdot v)$. Then, for all $x \in V'$, the map

$$\mathbb{R}^{n'} \xrightarrow{\tilde{\alpha}_x^{-1} \phi_x^{-1} \phi'_x} \mathbb{R}^n$$

is the standard inclusion. //

<u>Cor</u> If $E' \subset E$ is a subbdl, then there exists a (unique) v.b. E/E' s.t.

$$(E/E')_x = E_x / E'_x.$$

//

A euklidean vecter bdl. of rk. n is a fiber bdl. with fiber $\mathbb{R}^n$ and group $O(n)$. The coordinate functions give each fiber the structure of a real inner product space of rk. n. If X is (para-) cpt, then every real vecter bdl. over X can be given a euklidean structure:

$$[X, BO(n)] \xrightarrow{\sim} [X, BGL_n(\mathbb{R})]\ .$$

Let E be a euklidean bdl. and let $E' \subset E$ be a subbdl. Then there exists a subbdl. $E'^{\perp} \subset E$ s.t. for all $x \in X$,

$$(E'^{\perp})_x = (E'_x)^{\perp}\ ,$$

and moreover,

$$E'^{\perp} \xrightarrow{\sim} E/E'\ .$$

Hence, if X is (para-) cpt. and $E' \subset E$ subbdl., then there exist a (non-can.) iso. of vecter bdls. over X,

$$E \approx E' \oplus E/E'\ .$$

An oriented real v.b. of rk. n is a fiber bdl. with fiber $\mathbb{R}^n$ and grp. $SL_n(\mathbb{R})$. The coord. fcts. give each fiber the structure of an oriented real vector space of rk. n. This, in turn, gives a preferred generator of

$$u_x \in H^n(E_x, E_x \setminus \{0\}; \mathbb{Z}),$$

for all $x \in X$. We define u_x as follows. Let $\Delta^n \xrightarrow{f} E_x$ be an orientation pres. affine map s.t. $0 \in f(\Delta^n \setminus \partial\Delta^n)$. Then f defines $\bar{f} \in C_n(E_x, E_x \setminus \{0\})$, and

$$\tilde{u}_x = [\bar{f}] \in H_n(E_x, E_x \setminus \{0\}, \mathbb{Z})$$

is a generator. Then u_x is given by

$$\begin{array}{ccc} H^n(E_x, E_x \setminus \{0\}) & \xrightarrow[\sim]{\text{can}} & \mathrm{Hom}(H_n(E_x, E_x \setminus \{0\}), \mathbb{Z}) \\ u_x & \longmapsto & \tilde{u}_x^* \end{array}$$

$$\tilde{u}_x^*(\tilde{u}_x) = 1.$$

⌈

$$\begin{array}{ccc} H^1(\mathbb{R}, \mathbb{R} \setminus \{0\})^{\otimes n} & \xrightarrow[\sim]{\cup} & H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \\ u \otimes \cdots \otimes u & \longmapsto & (-1)^{[\frac{n}{2}]} u \end{array}$$

⌋

Let E be an or. real v.b. of rk. n over X, and let $E_0 \subset E$ be the compl. of the zero section. Then the comp.

$$H^*(X) \otimes H^*(E, E_0) \xrightarrow[\sim]{p^* \otimes \mathrm{id}} H^*(E) \otimes H^*(E, E_0)$$

$$\xrightarrow{\cup} H^*(E, E_0)$$

makes $H^*(E, E_0)$ a graded $H^*(B)$-mod.

<u>Thm</u> (Thom iso.) There exists a unique coh. class $u \in H^n(E, E_0)$ s.t.

$$\begin{array}{ccc} H^n(E, E_0) & \xrightarrow{i_x^*} & H^n(E_x, E_x \setminus \{0\}) \\ u & \longmapsto & u_x \end{array}$$

for all $x \in X$. Moreover, the graded $H^*(X)$-module $H^*(E, E_0)$ is free of rank 1 gen. by u.

 5/13/03

Let $E \xrightarrow{p} X$ be an orient. real v.b. of rk. n. We have defined the fund. coh. class $u_x \in H^n(E_x, E_x \setminus \{0\}; \mathbb{Z})$ assoc. w. the orientation of E.

Thm (Thom iso.) There exists a unique coh. class $u \in H^n(E, E_0)$ s.t. for all $x \in X$,

$$\begin{array}{ccc} H^n(E, E_0) & \xrightarrow{i_x^*} & H^n(E_x, E_x \setminus \{0\}) \\ u & \longmapsto & u_x \end{array}$$

Moreover, the map

$$\begin{array}{ccc} H^i(X) & \longrightarrow & H^{n+i}(E, E_0) \\ a & \longmapsto & p^*(a) \cup u \end{array}$$

is an iso., for all integers i.

Pf We prove the statement only for X cpt.; see [MS] for the general case. Suppose first that E is a trivial bdl., and pick an or. pres. trivialization

$$\begin{array}{ccc} X \times \mathbb{R}^n & \underset{\sim}{\xrightarrow{\phi}} & E \\ \downarrow pr_1 & & \downarrow p \\ X & = & X \end{array}$$

Then

$$H^{n+i}(E,E_0) \underset{\sim}{\xrightarrow{\phi^*}} H^{n+i}(X\times\mathbb{R}^n, X\times\mathbb{R}^n\setminus\{0\}) \underset{\sim}{\xleftarrow{\times}} H^i(X)\otimes H^n(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\})$$

so the class corresponding to $1\otimes u$ by this iso. does the job.

Suppose $X = X'\cup X''$ with $X', X''\subset X$ open and that the theorem holds for the restrictions E', E'', and $\bar{E}$ of E to X', X'', and $\bar{X} = X'\cap X''$. Consider the Mayer-Vietoris seq.

$$\cdots\to H^{n-1}(\bar{E},\bar{E}_0)\xrightarrow{\partial} H^n(E,E_0)\to \begin{matrix} H^n(E',E'_0)\\ \oplus \\ H^n(E'',E''_0)\end{matrix} \to H^n(\bar{E},\bar{E}_0)\to\cdots$$

By the uniqueness of $\bar{u}$, $u'\oplus u''$ maps to zero. Hence, there exists $u\in H^n(E,E_0)$ that restricts to u' and u''. This u is unique, since $H^{n-1}(\bar{E},\bar{E}_0)=0$. A 5-lemma arg. using the same seq. shows that

$$\begin{aligned} H^i(X) &\xrightarrow{\sim} H^{n+i}(E,E_0). \\ a &\longmapsto p^*(a)\cup u \end{aligned}$$

The proof for X cpt. is completed by easy induction. //

Consider

$$\cdots \to H^n(E, E_0) \xrightarrow{j^*} H^n(E) \xrightarrow{i^*} H^n(E_0) \to \cdots$$
$$\sim \uparrow p^*$$
$$H^n(X)$$

We define the Euler class

$$e = e(\mathcal{E}) \in H^n(X)$$

by $p^* e = j^* u$.

Prop (i) $e(f^*\mathcal{E}) = f^* e(\mathcal{E})$

(ii) $e(\bar{\mathcal{E}}) = - e(\mathcal{E})$

(iii) If $\operatorname{rk} \mathcal{E}$ is odd, then $2 e(\mathcal{E}) = 0$.

(iv) $e(\mathcal{E} \oplus \mathcal{E}') = e(\mathcal{E}) \cup e(\mathcal{E}')$.

(v) If $E \to X$ has a nowhere zero section, then $e(\mathcal{E}) = 0$.

pf (iii) The foll. iso.

$$\begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ \downarrow & & \downarrow \\ E & \longrightarrow & E \\ \downarrow & & \downarrow \\ X & = & X \end{array} \qquad \eta \mapsto -\eta$$

is orient. reversing if n is odd. So

$$e(\mathcal{E}) = e(\bar{\mathcal{E}}) = -e(\mathcal{E}).$$

Alternatively, by Thom Iso.

$$H^n(X) \xrightarrow{\sim} H^{2n}(E, E_0)$$

$$e \longmapsto p^*(e) \cup u$$

and

$$p^*(e) \cup u = j^*(u) \cup u = u \cup u = (-1)^{n \cdot n} u \cup u.$$

(iv) Let $n = \operatorname{rk} \mathcal{E}$ and $n' = \operatorname{rk} \mathcal{E}'$. Then

$$u(\mathcal{E} \times \mathcal{E}') = (-1)^{nn'} u(\mathcal{E}) \times u(\mathcal{E}') \Longrightarrow$$

$$e(\mathcal{E} \times \mathcal{E}') = (-1)^{nn'} e(\mathcal{E}) \times e(\mathcal{E}') \overset{(iii)}{=} e(\mathcal{E}) \times e(\mathcal{E}').$$

(v) Use the diagr.

$$\begin{tikzcd} \cdots \arrow[r] & H^n(E, E_0) \arrow[r, "j^*"] & H^n(E) \arrow[r, "i^*"] & H^n(E_0) \arrow[d, "\sigma^*", "\sim"'] \arrow[r] & \cdots \\ & & H^n(X) \arrow[u, "p^*", "\sim"'] \arrow[r, equal] & H^n(X) & \end{tikzcd}$$

//

Rem If M is a sm., cpt., or. mfd, then

$$\langle e(TM), [M] \rangle = \chi(M).$$

See [MS] for a proof. //

Consider again the pair seq.

$$\begin{tikzcd}
\cdots \arrow[r] & H^{n+i}(E, E_0) \arrow[r, "j^*"] & H^{n+i}(E) \arrow[r, "i^*"] & H^{n+i}(E_0) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^i(X) \arrow[r] \arrow[u, "\sim"', "p^*(a)\cup u"] & H^{n+i}(X) \arrow[r] \arrow[u, "\sim"', "p^*"] & H^{n+i}(E_0) \arrow[r] \arrow[u, equal] & \cdots
\end{tikzcd}$$

$$\overset{\uparrow}{a}$$

$$a \longmapsto a \cup e$$

The lower seq. is called the <u>Gysin</u> seq.

A complex vector space V has a canonical orientation regarded as a real vector space. Indeed, a $\mathbb{C}$-basis $v_1, \ldots, v_n$ (and a choice of square root of -1) determines an $\mathbb{R}$-basis $v_1, iv_1, \ldots, v_n, iv_n$, and a permutation of the elm. in the $\mathbb{C}$-basis induces an even perm. of the elm. of the $\mathbb{R}$-basis. It follows that a complex vector bdl. has a canonical orientation regarded as a real v.b.

The <u>tautological</u> <u>line</u> <u>bdl</u>. over $\mathbb{C}P^{n-1}$ is the cx. line bdle. $\mathcal{O}(-1)$ given by

$$E = \{ (L,v) \in \mathbb{C}P^{n-1} \times \mathbb{C}^n \mid v \in L \} \xrightarrow{pr_1} \mathbb{C}P^{n-1}.$$

Since a non-zero vector $v \in \mathbb{C}^n$ determines a unique line, we have a homeo.

$$E_0 \xrightarrow{\sim} \mathbb{C}^n \setminus \{0\} \simeq S^{2n-1}$$

Hence, the Gysin seq. shows that

$$H^*(\mathbb{C}P^n) = S_{\mathbb{Z}}\{e\} / (e^n) ,$$

where $e = e(\mathcal{O}(-1))$.

Let $E \xrightarrow{p} X$ be a complex vector bdl. of rk. n,

$$E = P \times_{GL_n(\mathbb{C})} \mathbb{C}^n .$$

We define the associated <u>projective</u> <u>bdl.</u> over X by

$$\mathbb{P}(E) = P \times_{GL_n(\mathbb{C})} \mathbb{C}P^{n-1} ,$$

where the $GL_n(\mathbb{C})$-action on $\mathbb{C}P^{n-1}$ is induced from that on $\mathbb{C}^n$. Consider the pull-back bdl.

$$\begin{array}{ccc} \pi^* E & \longrightarrow & E \\ \downarrow & & \downarrow p \\ \mathbb{P}(\mathcal{E}) & \xrightarrow{\pi} & X \end{array}.$$

The total space $\pi^* E$ consists of pairs (L, v), where $L \subset E_x$ and $v \in E_x$ is a line and a vector in the same fiber E_x. There is a tautological line bdl $\mathcal{O}(-1)$ over $\mathbb{P}(\mathcal{E})$ defined as the subbdl.

$$\mathcal{O}(-1) \subset \pi^* E$$

of pairs (L, v) s.t. $v \in L$. It restricts to the tautological bdl. on

$$\mathbb{P}(\mathcal{E})_x = \mathbb{P}(E_x).$$

An argument analogous to the pf. of the Thom Iso. thm. shows:

<u>Prop</u> (Projective bdl. formula) As a graded $H^*(X)$-module, $H^*(\mathbb{P}(\mathcal{E}))$ is free of rk. n gen. by $1, e, \dots, e^{n-1}$, where $e = e(\mathcal{O}(-1)) \in H^2(\mathbb{P}(\mathcal{E}))$. //

<u>Cor</u> (Splitting principle) Let $\mathcal{E}$ be a cx. v.b. over a cpt. space X. Then there exists $X' \xrightarrow{f} X$ with X' cpt. s.t.

(i) $f^*\mathcal{E}$ is iso. to a direct sum of cx. line bdls.

(ii) $H^*(X) \xrightarrow{f^*} H^*(X')$ is injective.

<u>pf</u> Let $X_1 = \mathbb{P}(\mathcal{E}) \xrightarrow{f_1} X$. Then

$$\mathcal{E}_1 := f_1^*\mathcal{E} \approx \mathcal{O}(-1) \oplus \mathcal{E}'$$

and $\operatorname{rk} \mathcal{E}' = \operatorname{rk} \mathcal{E} - 1$. The proj. bdl. formula shows that

$$H^*(X) \xrightarrow{f_1^*} H^*(X_1)$$

is injective. Induction. //

Let $\mathcal{E}$ be a cx. v.b. over X. We construct the Chern classes

$$c_i(\mathcal{E}) \in H^{2i}(X, \mathbb{Z})$$

which satisfy

(i) $c_i(f^*\mathcal{E}) = f^* c_i(\mathcal{E})$.

(ii) $c_i(\mathcal{E} \oplus \mathcal{E}') = \sum_{0 \le j \le i} c_j(\mathcal{E}) \cup c_{i-j}(\mathcal{E}')$

(iii) Let $\mathcal{O}(-1)$ be the tautological line bdl. over $\mathbb{CP}^\infty$. Then

$$c_i(\mathcal{O}(-1)) = \begin{cases} 1 & , i = 0, \\ -e(\mathcal{O}(-1)) & , i = 1, \\ 0 & , i > 1. \end{cases}$$

Prop The properties (i) – (iii) characterizes the classes $c_i(\mathcal{E})$ uniquely.

Pf Suppose that both $c_i(\mathcal{E})$ and $c_i'(\mathcal{E})$ satisfy (i) – (iii). Since every line bdl. (over a para-cpt. space) is of the form

$\mathcal{L} = f^*\mathcal{O}(-1)$, for some map $X \xrightarrow{f} \mathbb{C}P^\infty$, properties (i) and (iii) show that $c_i(\mathcal{L}) = c_i'(\mathcal{L})$. Property (ii) show that $c_i(\mathcal{E}) = c_i'(\mathcal{E})$, if $\mathcal{E}$ is a sum of line bdls. The general case follows from (i) and the splitting principle. //

Rem (i) $c_0(\mathcal{E}) = 1$ and $c_i(\mathcal{E}) = 0$, $i > \operatorname{rk} \mathcal{E}$.

(ii) If $c(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E}) + \cdots$ is the total Chern class, then (ii) can be rewritten as

$$c(\mathcal{E} \oplus \mathcal{E}') = c(\mathcal{E}) \cup c(\mathcal{E}') .$$

//

Let $\mathcal{E}$ be a cx. v.b. of rk. n over X, and let $\mathbb{P}(\mathcal{E})$ be the associated projective bdle. We recall that

$$H^*(\mathbb{P}(\mathcal{E})) = H^*(X)\{1, e, \ldots, e^{n-1}\},$$

where $e = e(\mathcal{O}(-1))$ is the Euler class of the tautological line bdl. over $\mathbb{P}(\mathcal{E})$. Suppose that X is cpt. s.t.

$$f^*\mathcal{E} \approx \mathcal{O}(-1) \oplus \mathcal{E}'.$$

If Chern classes have been defined, they must satisfy

$$f^* c(\mathcal{E}) = c(\mathcal{O}(-1)) \cdot c(\mathcal{E}')$$

$$= (1-e) \cdot c(\mathcal{E}').$$

Hence,

$$c(\mathcal{E}') = f^* c(\mathcal{E}) \cdot (1-e)^{-1}$$

$$= f^* c(\mathcal{E})(1 + e + e^2 + \ldots).$$

Since $\mathrm{rk}_{\mathbb{C}}\, \mathcal{E}' = n-1$, $c_n(\mathcal{E}') = 0$, so this equation gives

$$e^n + f^* c_1(\mathcal{E})\, e^{n-1} + \cdots + f^* c_n(\mathcal{E}) = 0,$$

and since $1, e, \ldots, e^{n-1}$ are lin. indept. over $H^*(X)$, this uniquely characterizes the classes $c_i(\mathcal{E})$. We take this as our definition, i.e. as a ring

$$H^*(\mathbb{P}(\mathcal{E})) = S_{\mathbb{Z}}\{e\} / (e^n + c_1(\mathcal{E})e^{n-1} + \cdots + c_n(\mathcal{E})),$$

where $e = e(\mathcal{O}(-1))$.

<u>Lemma</u> $c(\mathcal{E}' \oplus \mathcal{E}'') = c(\mathcal{E}') \cdot c(\mathcal{E}'')$.

<u>pf</u> let $\mathcal{E} = \mathcal{E}' \oplus \mathcal{E}'$ and consider the canonical inclusions

$$\mathbb{P}(\mathcal{E}') \xrightarrow{i'} \mathbb{P}(\mathcal{E}) \xleftarrow{i''} \mathbb{P}(\mathcal{E}'').$$

We have $i'^* \mathcal{O}(-1) = \mathcal{O}(-1)'$ and $i''^* \mathcal{O}(-1) = \mathcal{O}(-1)''$. Define $x', x'' \in H^*(\mathbb{P}(\mathcal{E}))$ by

$$x' = e^{n'} + f^* c_1(\mathcal{E}')\, e^{n'-1} + \cdots + f^* c_{n'}(\mathcal{E}'),$$

$$x'' = e^{n''} + f^* c_1(\mathcal{E}'')\, e^{n''-1} + \cdots + f^* c_{n''}(\mathcal{E}''),$$

where $n' = \mathrm{rk}_{\mathbb{C}} \mathcal{E}'$, $n'' = \mathrm{rk}_{\mathbb{C}} \mathcal{E}''$. Since the classes $c_i(-)$ are natural, we have $i'^*(x') = 0$ and $i''^*(x'') = 0$. Let

$$U' = \mathbb{P}(\mathcal{E}) \setminus \mathbb{P}(\mathcal{E}') \simeq \mathbb{P}(\mathcal{E}'')$$

$$U'' = \mathbb{P}(\mathcal{E}) \setminus \mathbb{P}(\mathcal{E}'') \simeq \mathbb{P}(\mathcal{E}')$$

By the pair sequences,

$$\begin{array}{ccc} H^{2n'}(\mathbb{P}(\mathcal{E}), U'') & \longrightarrow & H^{2n'}(\mathbb{P}(\mathcal{E})) \\ \Psi & & \Psi \\ \exists\, y' & \longmapsto & x' \end{array}$$

$$\begin{array}{ccc} H^{2n''}(\mathbb{P}(\mathcal{E}), U') & \longrightarrow & H^{2n''}(\mathbb{P}(\mathcal{E})) \\ \Psi & & \Psi \\ \exists\, y'' & \longmapsto & x'' \end{array}$$

It follows that

$$\begin{array}{ccc} H^{2n}(\mathbb{P}(\mathcal{E}), U' \cup U'') & \longrightarrow & H^{2n}(\mathbb{P}(\mathcal{E})) \\ \Psi & & \Psi \\ y' \cdot y'' & \longmapsto & x' \cdot x'' \end{array}$$

But $U' \cup U'' = \mathbb{P}(\mathcal{E})$, so the left hand group is zero. Hence $x' \cdot x'' = 0$, i.e.

$$(e^{n'} + f^* c_1(\mathcal{E}') e^{n'-1} + \cdots + f^* c_{n'}(\mathcal{E}'))$$
$$\cdot (e^{n''} + f^* c_1(\mathcal{E}'') e^{n''-1} + \cdots + f^* c_{n''}(\mathcal{E}'')) = 0 .$$

Compare with

$$e^n + f^* c_1(\mathcal{E}) + \cdots + f^* c_n(\mathcal{E}) = 0 . \qquad //$$

<u>Lemma</u> Let $\mathcal{L}$ and $\mathcal{L}'$ be line bdls. Then

$$c_1(\mathcal{L} \otimes \mathcal{L}') = c_1(\mathcal{L}) + c_1(\mathcal{L}') .$$

pf Line bdls (over a cpt. space) are classified by (htpy classes of) maps to $\mathbb{C}P^\infty$, and tensor product corresponds to a map

$$\begin{tikzcd} X \arrow[r, "(\mathcal{L}, \mathcal{L}')"] \arrow[rr, bend right, "\mathcal{L} \otimes \mathcal{L}'"'] & \mathbb{C}P^\infty \times \mathbb{C}P^\infty \arrow[r, "\otimes"] & \mathbb{C}P^\infty \end{tikzcd}$$

Let $a = c_1(\mathcal{O}(-1))$. Then

$$H^*(\mathbb{C}P^\infty) = S_{\mathbb{Z}}\{a\}$$

$$H^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = S_{\mathbb{Z}}\{a \times 1, 1 \times a\} .$$

The following diagr. comm. (up to htpy)

$$\begin{tikzcd}
\mathbb{C}P^\infty \arrow[d, "{(\mathcal{O}(-1),\, \varepsilon^1)}"'] \arrow[dr, "\mathcal{O}(-1)"] & \\
\mathbb{C}P^\infty \times \mathbb{C}P^\infty \arrow[r, "\otimes"] & \mathbb{C}P^\infty \\
\mathbb{C}P^\infty \arrow[u, "{(\varepsilon^1,\, \mathcal{O}(-1))}"] \arrow[ur, "\mathcal{O}(-1)"'] &
\end{tikzcd}$$

On coh. this becomes

$$\begin{tikzcd}
a & 0 & S_{\mathbb{Z}}\{a\} & \\
a\times 1 \arrow[u, mapsto] \arrow[d, mapsto] & 1\times a \arrow[u, mapsto] \arrow[d, mapsto] & S_{\mathbb{Z}}\{a\times 1, 1\times a\} \arrow[u] \arrow[d] & S_{\mathbb{Z}}\{a\} \arrow[l, "\otimes^*"'] \arrow[ul, equal] \arrow[dl, equal] \\
0 & a & S_{\mathbb{Z}}\{a\} &
\end{tikzcd}$$

which shows that

$$\begin{tikzcd}
S_{\mathbb{Z}}\{a\times 1, 1\times a\} & S_{\mathbb{Z}}\{a\} \arrow[l, "\otimes^*"'] \\
a\times 1 + 1\times a & a \arrow[l, mapsto]
\end{tikzcd}$$

Thus

$$H^*(X) \xleftarrow{(\mathcal{L},\mathcal{L}')^*} H^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \xleftarrow{\otimes^*} H^*(\mathbb{C}P^\infty)$$

$$c_1(\mathcal{L}) + c_1(\mathcal{L}') \leftarrow\!\shortmid c_1(\mathcal{O}(-1))\times 1 + 1 \times c_1(\mathcal{O}(-1)) \leftarrow\!\shortmid c_1(\mathcal{O}(-1))$$

But, by naturality, the image of $c_1(\mathcal{O}(-1))$ is equal to $c_1(\mathcal{L}\otimes\mathcal{L}')$. //

<u>Cor</u> $c_k(\mathcal{E}^*) = (-1)^k c_k(\mathcal{E})$.

<u>pf</u> If $\mathcal{L}$ is a line bdle. Then

$$\mathcal{L}^* \otimes \mathcal{L} \xrightarrow{\sim} \varepsilon^1$$

so, by the lemma, $c_1(\mathcal{L}^*) + c_1(\mathcal{L}) = 0$.
By the splitting principle, we can assume

$$\mathcal{E} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n.$$

Then

$$\mathcal{E}^* = \mathcal{L}_1^* \oplus \cdots \oplus \mathcal{L}_n^*$$

so

$$c_k(\mathcal{E}^*) = c_k(\mathcal{L}_1^* \oplus \cdots \oplus \mathcal{L}_n^*)$$

$$= \sigma_k(c_1(\mathcal{L}_1^*), \ldots, c_1(\mathcal{L}_n^*))$$

$$= \sigma_k(-c_1(\mathcal{L}_1), \ldots, -c_1(\mathcal{L}_n))$$

$$= (-1)^k \sigma_k(c_1(\mathcal{L}_1), \ldots, c_1(\mathcal{L}_n))$$

$$= (-1)^k c_k(\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n)$$

$$= (-1)^k c_k(\mathcal{E}).$$

Here $\sigma_k(x_1, \ldots, x_n)$ is the k'th elem. symmetric polynomial. //

Let $\mathcal{O}(-1)$ be the tautological line bdl. over $\mathbb{C}P^n$, and recall that $\mathcal{O}(-1)$ is a subbdl. of the trivial bdl. ε^{n+1}; let $\mathcal{O}(-1)^\perp \subset \varepsilon^{n+1}$ be the compl. w.r.t. the hermetian metric on ε^{n+1}. We recall that $\mathbb{C}P^n$ is a complex mfd. If $L \in \mathbb{C}P^n$, then

$$U_L = \{ L' \in \mathbb{C}P^n \mid L' \cap L^\perp = \{0\} \}$$

is an open nbhd. of L, and

$$\begin{aligned} \mathrm{Hom}_{\mathbb{C}}(L, L^\perp) &\xrightarrow[\sim]{\Gamma} U_L \\ f &\longmapsto \Gamma_f \end{aligned}$$

is a chart around L. This gives rise to an iso. of cx. v.b. over $\mathbb{C}P^n$,

$$\mathrm{Hom}(\mathcal{O}(-1), \mathcal{O}(-1)^\perp) \xrightarrow{\sim} T\mathbb{C}P^n ,$$

which on the fiber over $L \in \mathbb{C}P^n$ is given by the map

$$\mathrm{Hom}(L, L^\perp) \xrightarrow{\sim} T_L \mathbb{C}P^n$$

$$f \longmapsto (D_0 \Gamma)(f) .$$

We use this to show:

Thm $c(T\mathbb{C}P^n) = c(\mathcal{O}(1))^{n+1}$.

Pf Here $\mathcal{O}(1) = \mathcal{O}(-1)^*$, so we have

$$T\mathbb{C}P^n \approx \mathcal{O}(1) \otimes \mathcal{O}(-1)^\perp .$$

Adding the trivial bdl. $\varepsilon^1 = \mathcal{O}(1) \otimes \mathcal{O}(-1)$ on both sides, we get

$$T\mathbb{C}P^n \oplus \varepsilon_1 \approx \mathcal{O}(1) \otimes \mathcal{O}(-1)^\perp \oplus \mathcal{O}(1) \otimes \mathcal{O}(-1)$$

$$\approx \mathcal{O}(1) \otimes \varepsilon^{n+1} \approx \underbrace{\mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(1)}_{n+1} .$$

Hence

$$c(T\mathbb{CP}^n) = c(T\mathbb{CP}^n \oplus \varepsilon^1)$$

$$= c(\underbrace{\mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(1)}_{n+1})$$

$$= \underbrace{c(\mathcal{O}(1)) \cdot \ldots \cdot c(\mathcal{O}(1))}_{n+1} . \qquad //$$

Finally, we show that

$$H^*(BU(n)) \xrightarrow{\sim} H^*(BT^n)^{\Sigma_n} .$$

The map is injective by the splitting principle. It lands in the fixed set of the Weyl grp. $N_{U(n)}(T^n)/T^n \approx \Sigma_n$ because ...

It is surjective by Newton's thm.

$$S_{\mathbb{Z}}\{\sigma_1, \ldots, \sigma_n\} \xrightarrow{\sim} S_{\mathbb{Z}}\{x_1, \ldots, x_n\}^{\Sigma_n} .$$

We conclude that every characteristic class of cx. v.b. of rk. n is a polynomial in the Chern classes.