18.906: Problem Set 1

Due: Thursday, February 20.

All chain complexes are concentrated in non-negative degrees.

1. Let k be a commutative ring, and let C_* be a chain complex of free k-modules,

$$C_0 \xleftarrow{d} C_1 \xleftarrow{d} C_2 \leftarrow \cdots .$$

Show that there is a chain homotopy between 0 and id if and only if $H_*(C_*) = 0$.

2. If $f\colon X \to Y$ is a chain map, we define the *mapping cone* C_f to be the complex

$$(C_f)_n = Y_n \oplus X_{n-1}, \qquad d(y,x) = (dy - f(x), -dx),$$

and the suspension ΣX to be the cokernel of the inclusion $\iota\colon Y \to C_f$ of the first summand, that is,

$$(\Sigma X)_n = X_{n-1}, \qquad d_{\Sigma X}(x) = -d_X(x).$$

Then we have a short exact sequence of complexes

$$0 \to Y \xrightarrow{\iota} C_f \xrightarrow{\mathrm{pr}} \Sigma X \to 0,$$

where pr is the canonical projection. Show that the connecting homomorphism

$$H_{n-1}(X) = H_n(\Sigma X) \xrightarrow{\partial} H_{n-1}(Y)$$

is equal to $-H_{n-1}(f)$.

3. Prove the following result of J. H. C. Whitehead. Let $f\colon C_* \to C'_*$ be a chain map between chain complexes of of free k-modules. Then f is a chain homotopy equivalence if and only if $H_*(f)$ is an isomorphism. (*Hint*: use 1. and 2.)

4. We consider the following two functors from the category of pairs of spaces and pairs of continuous maps to the category of chain complexes and chain maps.

$$F(X,Y) = C_*(X \times Y),$$
$$G(X,Y) = C_*(X) \otimes C_*(Y).$$

We define a new chain complex denoted $\mathrm{Hom}(F,G)$ as follows. Let $\mathrm{Hom}(F,G)_n$ be the set of natural maps of graded abelian groups of degree n from F to G. That is, an element $f \in \mathrm{Hom}(F,G)_n$ is a sequence $f = (f_r)_{r\in\mathbb{Z}}$ of natural maps of abelian groups $f_r\colon F(X,Y)_r \to G(X,Y)_{r+n}$, but we do *not* assume that f is a chain map. The set $\mathrm{Hom}(F,G)_n$ is an abelian group with the sum is given by $(f+g)(\sigma) = f(\sigma) + g(\sigma)$. The differential

$$d\colon \mathrm{Hom}(F,G)_n \to \mathrm{Hom}(F,G)_{n-1}$$

is defined by the equation

$$d(f(\sigma)) = (df)(\sigma) + (-1)^{\deg(f)} f(d\sigma).$$

Here $\mathrm{Hom}(F,G)_{-1} = 0$, by definition.

(i) Let $f = (f_r)_{r\in\mathbb{Z}} \in \mathrm{Hom}(F,G)_0$. Show that there exists an integer $\epsilon(f)$ such that for all (X,Y), the following diagram commutes.

$$\begin{array}{ccc} F(X,Y)_0 & \xrightarrow{f_0} & G(X,Y)_0 \\ \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon\otimes\epsilon} \\ \mathbb{Z} & \xrightarrow{1\mapsto\epsilon(f)(1\otimes 1)} & \mathbb{Z}\otimes\mathbb{Z}. \end{array}$$

Show further that the assignment $f \mapsto \epsilon(f)$ is a chain map

$$\epsilon\colon \mathrm{Hom}(F,G) \to \mathbb{Z}.$$

(ii) Show that the map $\epsilon\colon \mathrm{Hom}(F,G) \to \mathbb{Z}$ is a quasi-isomorphism, i.e. that the induced map on homology

$$\epsilon_*\colon H_*(\mathrm{Hom}(F,G)) \to \mathbb{Z}$$

is an isomorphism. (This means, in particular, that $H_n(\mathrm{Hom}(F,G))$ is equal to zero, if $n > 0$.)