18.906: Problem Set 3

Due: Thursday, March 6.

1. Let K and K' be simplicial complexes with vertex sets V and V', respectively, and let $f\colon |K| \to |K'|$ be a continuous map. Show that f has a simplicial approximation if and only if for all $v \in V$, there exists $v' \in V'$ such that

$$f(\mathrm{st}(v)) \subset \mathrm{st}(v').$$

(*Hint*: For all $v \in V$, choose $v' = \varphi(v) \in V$ such that $f(\mathrm{st}(v)) \subset \mathrm{st}(v')$. Show that the map $\varphi\colon V \to V'$ is a simplicial map and a simplicial approximation to f.)

2. Let $f\colon Y \to X$ be a map and consider the following pull-back diagram.

$$\begin{array}{ccc} X & \xleftarrow{\;f\;} & Y \\ \uparrow{\scriptstyle \mathrm{ev}_0} & & \uparrow{\scriptstyle p_0} \\ X^{[0,1]} & \xleftarrow{\;q\;} & E_f. \end{array}$$

The identity map on Y and the map $Y \to X^{[0,1]}$ given as the composite of $f\colon Y \to X$ and the map $c\colon X \to X^{[0,1]}$, which takes x to the constant path at x, give rise to a map $i\colon Y \to E_f$. We also let $p_1\colon E_f \to X$ be the composite of $q\colon E_f \to X^{[0,1]}$ and $\mathrm{ev}_1\colon X^{[0,1]} \to X$. We then have the following commutative diagram.

$$\begin{array}{ccccc} X & \xleftarrow{\;p_1\;} & E_f & \xrightarrow{\;p_0\;} & Y \\ & {\scriptstyle f}\nwarrow & \uparrow{\scriptstyle i} & /\!\!/ & \\ & & Y & & \end{array}$$

(i) Show that p_1 is a Hurewicz fibration.

(ii) Show further that the following composite is homotopic to the identify.

$$E_f \xrightarrow{p_0} Y \xrightarrow{i} E_f.$$

We define the *homotopy fiber* of f at $x_0 \in X$ by the following pull-back diagram.

$$\begin{array}{ccc} X & \xleftarrow{\;p_1\;} & E_f \\ \uparrow{\scriptstyle x_0} & & \uparrow \\ * & \longleftarrow & F_f. \end{array}$$

Let $y_0 \in f^{-1}(x_0)$, and let $z_0 \in F_f$ be the point such that $p_0(z_0) = y_0$ and such that $q(z_0)$ is the constant path at x_0.

(iii) Conclude that there is a long-exact sequence of pointed sets

$$\dots \to \pi_n(F_f, z_0) \xrightarrow{p_{0*}} \pi_n(Y, y_0) \xrightarrow{f_*} \pi_n(X, x_0) \xrightarrow{\partial} \pi_{n-1}(F_f, z_0) \to \dots$$