## 18.906: Problem Set 3

Due: Thursday, March 6.

**1.** Let K and K' be simplicial complexes with vertex sets V and V', respectively, and let  $f: |K| \to |K'|$  be a continuous map. Show that f has a simplicial approximation if and only if for all  $v \in V$ , there exists  $v' \in V'$  such that

$$f(\operatorname{st}(v)) \subset \operatorname{st}(v')$$
.

(*Hint*: For all  $v \in V$ , choose  $v' = \varphi(v) \in V$  such that  $f(\operatorname{st}(v)) \subset \operatorname{st}(v')$ . Show that the map  $\varphi \colon V \to V'$  is a simplicial map and a simplicial approximation to f.)

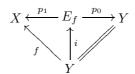
**2.** Let  $f: Y \to X$  be a map and consider the following pull-back diagram.

$$X \leftarrow f \qquad Y$$

$$\uparrow_{\text{ev}_0} \qquad \uparrow_{p_0}$$

$$X^{[0,1]} \leftarrow E_f.$$

The identity map on Y and the map  $Y \to X^{[0,1]}$  given as the composite of  $f\colon Y \to X$  and the map  $c\colon X \to X^{[0,1]}$ , which takes x to the constant path at x, give rise to a map  $i\colon Y \to E_f$ . We also let  $p_1\colon E_f \to X$  be the composite of  $q\colon E_f \to X^{[0,1]}$  and  $\operatorname{ev}_1\colon X^{[0,1]} \to X$ . We then have the following commutative diagram.



- (i) Show that  $p_1$  is a Hurewicz fibration.
- (ii) Show further that the following composite is homotopic to the identify.

$$E_f \xrightarrow{p_0} Y \xrightarrow{i} E_f$$
.

We define the homotopy fiber of f at  $x_0 \in X$  by the following pull-back diagram.

$$\begin{array}{ccc}
X & \xrightarrow{p_1} E_f \\
\uparrow^{x_0} & \uparrow \\
* & \longleftarrow F_f.
\end{array}$$

Let  $y_0 \in f^{-1}(x_0)$ , and let  $z_0 \in F_f$  be the point such that  $p_0(z_0) = y_0$  and such that  $q(z_0)$  is the constant path at  $x_0$ .

(iii) Conclude that there is a long-exact sequence of pointed sets

$$\ldots \to \pi_n(F_f, z_0) \xrightarrow{p_{0*}} \pi_n(Y, y_0) \xrightarrow{f_*} \pi_n(X, x_0) \xrightarrow{\partial} \pi_{n-1}(F_f, z_0) \to \ldots$$

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