18.906: Problem Set 4

Due: Thursday, March 13.

1. Let (X, A) be a pair of locally compact spaces, and let Y be any space. Suppose that (X, A) has the homotopy extension property with respect to all spaces. Show that the map induced by the inclusion

$$Y^X \xrightarrow{i^*} Y^A$$

has the homotopy lifting property with respect to all spaces, *i.e.* that i^* is a Hurewicz fibration.

2. Let $f\colon C \to D$ and $g\colon D \to C$ be functors. We say that (f, g) is an adjoint pair (or that f is left adjoint to g or that g is right adjoint to f) if there exists a natural (with respect to maps in both variables) bijection

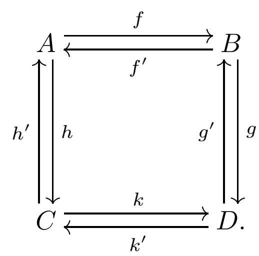
$$\varphi\colon \operatorname{Hom}_D(f(c), d) \xrightarrow{\sim} \operatorname{Hom}_C(c, g(d)).$$

The natural transformation $\eta\colon c \to g(f(c))$ given by $\eta = \eta_c = \varphi(\mathrm{id}_{f(c)})$ is called the *unit* of the adjunction. The *counit* is the natural transformation $\epsilon\colon f(g(d)) \to d$ given by $\epsilon = \epsilon_d = \varphi^{-1}(\mathrm{id}_{g(d)})$.

(i) Let $a\colon f(c) \to d$ and $b\colon c \to g(d)$ be morphisms in D and C, respectively. Show that the morphisms $\varphi(a)\colon c \to g(d)$ and $\varphi^{-1}(b)\colon f(c) \to d$ are given by the following composites, respectively.

$$c \xrightarrow{\eta_c} g(f(c)) \xrightarrow{g(a)} g(d),$$
$$f(c) \xrightarrow{f(b)} f(g(d)) \xrightarrow{\epsilon_d} d.$$

(ii) Consider the following diagram of categories and functors.



Suppose that f (resp. g, resp. h, resp. k) is left adjoint to f' (resp. g', resp. h', resp. k'). Show that there is a natural isomorphism from gf to kh if and only if there is a natural isomoprhism from $f'g'$ to $h'k'$.

3. Let $f\colon A \to B$ be a ring homomorphism, let M be a (right) A-module, and let N be a (left) B-module. The ring homomorphism f allows us to view N as a (left) A-module. Show that there is a spectral sequence

$$E^2_{s,t} = \operatorname{Tor}^B_s(\operatorname{Tor}^A_t(M, B), N) \Rightarrow \operatorname{Tor}^A_{s+t}(M, N).$$

(*Hint*: Choose a resolution $P_* \to M$ by projective (right) A-modules and a resolution $Q_* \to N$ by projective (left) B-modules. Then consider the two spectral sequences associated with the bi-complex $P_* \otimes_A Q_* \approx (P_* \otimes_A B) \otimes_B Q_*$.)