18.906: Problem Set 5

Due: Thursday, April 10.

Let $f\colon E \to B$ be a Serre fibration with B simply-connected. This problem set concerns the cohomological Serre spectral sequence. It takes the form

$$E_2^{s,t} = H^s(B, H^t(F)) \Rightarrow H^{s+t}(E),$$

where $F = F_b = f^{-1}(b)$ is the fiber over $b \in B$. (We recall that the cohomology of the fibers F_b are canonically isomorphic.) The upper indexing indicates that the spectral sequence is *cohomology type*, so the differentials are maps

$$d_r\colon E_r^{s,t} \to E_r^{s+r,t-r+1}.$$

This spectral sequence is *multiplicative*. Hence, (E_r, d_r) is an anti-commutative differential-bigraded-ring, which means that there is a product

$$E_r^{s,t} \otimes E_r^{s',t'} \xrightarrow{\cup} E_r^{s+s',t+t'}$$

which satisfies

$$\omega \cup \omega' = (-1)^{\deg \omega \cdot \deg \omega'} \omega' \cup \omega,$$
$$d_r(\omega \cup \omega') = d_r(\omega) \cup \omega' + (-1)^{\deg \omega} \omega \cup d_r(\omega'),$$

where deg is the total degree. In addition, the isomorphism $E_{r+1} \approx H^*(E_r, d_r)$ is multiplicative.

There is an isomorphism of (anti-commutative) bigraded rings

$$E_2^{*,*} = H^*(B, H^*(F))$$

with the product on the right given by the cup-product. Finally, the spectral sequence converges to $H^*(E)$ as an algebra. This meaning of this is as follows. Firstly, there is, associated with the spectral sequence, a natural descending filtration by graded ideals

$$H^*(E) = \mathrm{Fil}^0 H^*(E) \supset \mathrm{Fil}^1 H^*(E) \supset \mathrm{Fil}^2 H^*(E) \supset \dots$$

and the cup-product restricts to maps

$$\mathrm{Fil}^s H^*(E) \otimes \mathrm{Fil}^{s'} H^*(E) \xrightarrow{\cup} \mathrm{Fil}^{s+s'} H^*(E).$$

It follows that the cup-product on $H^*(E)$ induces a product

$$\mathrm{gr}^s H^i(E) \otimes \mathrm{gr}^{s'} H^{i'}(E) \xrightarrow{\cup} \mathrm{gr}^{s+s'} H^{i+i'}(E)$$

which makes $\mathrm{gr}^* H^*(E)$ an anti-commutative bi-graded ring. Secondly, there is a natural isomorphism of anti-commutative bigraded rings

$$E_\infty^{*,*} \approx \mathrm{gr}^* H^*(E).$$

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1. Consider the cohomological Serre spectral sequence for the path-space fibration associates with the n-sphere with $n \geq 3$ odd,

$$\Omega(S^n) \to P(S^n) \to S^n. \qquad \text{(over)}$$

(i) Show that

$$H^i(\Omega(S^n)) \approx \begin{cases} \mathbb{Z} & \text{if } n-1 \text{ divides } i, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) Let $x \in H^{n-1}(\Omega(S^n))$ be a generator. Show that the m-fold cup-product of x,

$$x^m \in H^{m(n-1)}(\Omega(S^n)),$$

is equal to $m!$ times a generator.

(iii) Let $x^{[m]} \in H^{m(n-1)}(\Omega(S^n))$ be the generator with $x^m = m! \cdot x^{[m]}$. Conclude that

$$x^{[m]} \cup x^{[m']} = \binom{m+m'}{m} x^{[m+m']}.$$

(We say that the cohomology ring $H^*(\Omega(S^n))$ is a *divided power algebra* on the generator x of degree $n-1$ and write $H^*(\Omega(S^n)) = \Gamma_{\mathbb{Z}}\{x\}$.)

2. Let $f\colon E \to B$ be a Serre fibration with B simply-connected and suppose that the fiber $F = F_b$ is a cohomology $(n-1)$-sphere with $n \geq 2$. This means that

$$H^i(F) \approx \begin{cases} \mathbb{Z} & \text{if } i = 0 \text{ or } n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $x \in H^{n-1}(F)$ be a generator, and let $y = d_n(x) \in H^n(B)$ be the image of x by the d_n-differential in the Serre spectral sequence. Show that there is a long-exact sequence of cohomology groups

$$\dots \to H^{i-1}(E) \to H^{i-(n-1)}(B) \xrightarrow{-\cup y} H^i(B) \xrightarrow{f^*} H^i(E) \to \dots.$$

3. Let $f\colon E \to B$ be a Serre fibration with B simply-connected. Suppose that the cohomology ring of the fiber $F = F_b$ is an exterior algebra on two generators x_1 and x_3 of degrees 1 and 3, respectively,

$$H^*(F) = \Lambda_{\mathbb{Z}}\{x_1, x_3\},$$

and suppose in addition that E is contractible. Show that the cohomology ring of B is a polynomial algebra on two generators y_2 and y_4 of degrees 2 and 4, respectively,

$$H^*(B) = S_{\mathbb{Z}}\{x_2, x_4\}.$$

(*Hint*: Write down an abstract spectral sequence with

$$E_2 = S_{\mathbb{Z}}\{x_2, x_4\} \otimes \Lambda_{\mathbb{Z}}\{y_1, y_3\}$$

and with $E_\infty = \mathbb{Z}$. Show that there is a map of spectral sequences from this abstract spectral sequence to the Serre spectral sequence. Use the comparison theorem.)