18.906: Problem Set 6

Due: Thursday, April 17.

This problem set calculates the cohomology ring $H^*(\mathbb{R}P^\infty, \mathbb{F}_2)$ by hand.

Let $k[G]$ be the group algebra of the group G over the commutative ring k, and let M be a left $k[G]$-module. (Note that to give a left $k[G]$-module structure on a k-module M is the same as giving a left action by G on M through k-linear maps.) The group cohomology of G with coefficients in M is defined by

$$H^*(G, M) = H^*(\mathrm{Hom}_{k[G]}(P_*, M)),$$

where $P_* \xrightarrow{\epsilon} k$ is a resolution of k by projective left $k[G]$-modules. Since any two such resolutions are chain-homotopy equivalent by a unique chain-homotopy class of chain maps, the cohomology groups are well-defined up to canonical isomorphism.

If M and N are two left $k[G]$-modules, then we view $M \otimes_k N$ as a left $k[G]$-module with the diagonal G-action: $g \cdot (m \otimes n) = (gm) \otimes (gn)$. We shall now define the cross-product in group cohomology,

$$H^*(G, M) \otimes_k H^*(G, N) \xrightarrow{\times} H^*(G, M \otimes_k N).$$

The Künneth formula shows that if $P_* \xrightarrow{\epsilon} k$ is a resolution of k by projective left $k[G]$-modules, then $P_* \otimes_k P_* \xrightarrow{\epsilon \otimes \epsilon} k \otimes_k k$ is a resolution of $k \otimes_k k$ by (projective) left $k[G]$-modules. Therefore, by the fundamental theorem of homological algebra, there exists a lift of the canonical isomorphism $k \xrightarrow{\sim} k \otimes k$ to a chain map α of complexes of left $k[G]$-modules

$$\begin{array}{ccc} P_* & \xrightarrow{\alpha} & P_* \otimes_k P_* \\ \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon \otimes \epsilon} \\ k & \xrightarrow{\sim} & k \otimes_k k, \end{array}$$

and any two such lifts are chain-homotopic. The cross-product is now the map on cohomology induced by the cochain map

$$\mathrm{Hom}_{k[G]}(P_*, M) \otimes \mathrm{Hom}_{k[G]}(P_*, N) \to \mathrm{Hom}_{k[G]}(P_*, M \otimes_k N)$$

given as the composite of the canonical map

$$\mathrm{Hom}_{k[G]}(P_*, M) \otimes \mathrm{Hom}_{k[G]}(P_*, N) \xrightarrow{\text{can}} \mathrm{Hom}_{k[G]}(P_* \otimes_k P_*, M \otimes_k N)$$

and the map induced from α. Finally, if $M = N = k$, the cross-product and the canonical isomorphism $k \otimes_k k \xrightarrow{\sim} k$ defines the cup-product in group cohomology

$$H^*(G, k) \otimes_k H^*(G, k) \xrightarrow{\cup} H^*(G, k).$$

This makes $H^*(G, k)$ an anti-commutative graded k-algebra.

1. Let G be a group, and let EG be a contractible space on which G acts properly discontinuously from the left. Then the canonical projection

$$EG \xrightarrow{f} BG = G \backslash EG$$

is a normal covering space, and G is canonically isomorphic to the group of deck transformations. Let k be a commutative ring, and let $k[G]$ be the group algebra. The left action by G on EG gives rise to a left $k[G]$-module structure on $C_*(EG, k)$.

(i) Show that $C_*(EG, k) \xrightarrow{\epsilon} k$ is a resolution of k by free left $k[G]$-modules.

(ii) Show that the map f induces an isomorphism

$$\mathrm{Hom}_{k[G]}(C_*(EG,k),k) \xrightarrow{\sim} \mathrm{Hom}_{k[G]}(C_*(BG,k),k) = \mathrm{Hom}_k(C_*(BG,k),k)$$

and conclude thereby that the group cohomology groups $H^*(G, k)$ and the singular cohomology groups $H^*(BG, k)$ are canonically isomorphic.

(iii) Show that the cup-products on $H^*(G, k)$ and $H^*(BG, k)$ agree.

2. Let $G = \langle T \rangle$ be the cyclic group of order two, and let k be a commutative ring. We let $W_* \xrightarrow{\epsilon} k$ be the standard resolution of k, with W_i a free $k[G]$-module on a generator e_i, and with the differential and augmentation given by

$$d(e_i) = \begin{cases} (T+1)e_{i-1} & \text{if } i \text{ is even,} \\ (T-1)e_{i-1} & \text{if } i \text{ is odd,} \end{cases}$$
$$\epsilon(e_0) = 1.$$

(i) Let $e_i^*\colon W_i \to k$ be the unique $k[G]$-linear map with $e_i^*(e_i) = 1$. Show that $\mathrm{Hom}_{k[G]}(W_i, k) = k \cdot e_i^*$ and that $d(e_i^*)$ is equal to 0 and $2e_{i+1}^*$, respectively, as i is even and odd.

(ii) Show that the formula

$$\alpha(e_i) = \sum_{0 \le s \le i} e_s \otimes T^{s+1} e_{i-s}$$

defines a lift of the canonical isomorphism $k \xrightarrow{\sim} k \otimes k$ to a chain map α of complexes of left $k[G]$-modules

$$\begin{array}{ccc} W_* & \xrightarrow{\alpha} & W_* \otimes_k W_* \\ \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon \otimes \epsilon} \\ k & \xrightarrow{\sim} & k \otimes_k k. \end{array}$$

(iii) Suppose that $2 = 0$ in k. Show that

$$H^*(G, k) = S_k\{x\},$$

where x is the cohomology class of e_1^*.

3. Show that $H^*(\mathbb{R}P^\infty, \mathbb{F}_2)$ is a polynomial algebra on a generator of degree one.