18.906: Problem Set 7

Due: Thursday, April 24.

The purpose of this problem set is to calculate $\pi_*(S^n) \otimes \mathbb{Q}$ for all $n \geq 0$. We shall use the following statement, which follows from the proof of the Hurewicz theorem given in class. Let X be a simply-connected space, and suppose that $H_i(X, \mathbb{Q})$ is equal to zero, for all $i < n$. Then the Hurewicz map induces an isomorphism

$$\pi_i(X) \otimes \mathbb{Q} \xrightarrow{\sim} H_i(X, \mathbb{Q}),$$

for all $i \leq n$.

1. Calculate the cohomology ring $H^*(K(\mathbb{Z}, n), \mathbb{Q})$, for all $n \geq 0$.

2. Let F_n be the homotopy fiber of a map $S^n \xrightarrow{f} K(\mathbb{Z}, n)$, which represents a generator of $H^n(S^n, \mathbb{Z})$. Then there is a Serre fibration

$$F_n \to E_f \to K(\mathbb{Z}, n)$$

with E_f homotopy equivalent to S^n.

(i) Show that

$$H^*(F_n, \mathbb{Q}) \approx \begin{cases} \Lambda_{\mathbb{Q}}\{\iota_{2n-1}\} & \text{if } n \text{ is even,} \\ \mathbb{Q} & \text{if } n \text{ is odd.} \end{cases}$$

(ii) Conclude that if n is odd, then $\pi_i(F_n) \otimes \mathbb{Q}$ is zero, for all $i \geq 0$, and that if n is even, then $\pi_i(F_n) \otimes \mathbb{Q}$ is zero, for $i < 2n - 1$, and isomorphic to $\mathbb{Q}$, for $i = 2n - 1$.

If n is even, we choose a map $F_n \xrightarrow{g} K(\mathbb{Z}, 2n-1)$ that represents a generator of $H^{2n-1}(F_n, \mathbb{Z}) \otimes \mathbb{Q} \xrightarrow{\sim} H^{2n-1}(F_n, \mathbb{Q})$. Let F'_n be the homotopy fiber of g. Then there is a Serre fibration

$$F'_n \to E_g \to K(\mathbb{Z}, 2n-1)$$

with E_g homotopy equivalent to F_n.

(iii) Show that $H^*(F'_n, \mathbb{Q}) = \mathbb{Q}$, and conclude that $\pi_*(F'_n) \otimes \mathbb{Q}$ is zero.

(iv) Calculate $\pi_i(S^n) \otimes \mathbb{Q}$ for all $n \geq 0$ and all $i \geq 0$.