18.906: Problem Set 9

Due: No.

In this problem, cohomology will be with $\mathbb{F}_2$-coefficients.

1. Let $E \xrightarrow{p} X$ be a real vector bundle of rank n over a compact space X. Show that there exists a unique cohomology class $u \in H^n(E, E_0)$ that restricts to the unique generator $u_x \in H^n(E_x, E_x \smallsetminus \{0\})$, for all $x \in X$. Show that as an $H^*(X)$-module, $H^*(E, E_0)$ is free of rank one generated by u.

2. Show that for a real vector bundle $\mathcal{E}$ over a compact space X, there exists characteristic classes $w_i(\mathcal{E}) \in H^i(X)$, $i \geq 0$, that are uniquely characterized by the following three properties.

(i) $w_i(f^*\mathcal{E}) = f^* w_i(\mathcal{E})$.

(ii) $w_i(\mathcal{E} \oplus \mathcal{E}') = \sum_{0 \leq j \leq i} w_j(\mathcal{E}) \cup w_{i-j}(\mathcal{E}')$.

(iii) Let $\mathcal{O}(-1)$ be the tautological line bundle over $\mathbb{R}P^n$. Then $w_i(\mathcal{O}(-1))$ is non-zero, for $0 \leq i \leq 1$, and zero, for $i > 1$.

3. Let $\mathcal{E}$ be a real vector bundle of rank n over X. Show that the composite

$$X \xrightarrow{f} BGL_n(\mathbb{R}) \xrightarrow{B\det} B\mathbb{R}^*,$$

where f classifies $\mathcal{E}$, represents $w_1(\mathcal{E}) \in H^1(X)$. Conclude that $\mathcal{E}$ is orientable if and only if $w_1(\mathcal{E}) = 0$.

4. Let $\phi \colon H^*(X) \xrightarrow{\sim} H^{*+n}(E, E_0)$ be the isomorphism $\phi(a) = p^*(a) \cup u$. Show that the classes $w_i'(\mathcal{E}) = (\phi^{-1} \operatorname{Sq}^i \phi)(a)$ satisfy the axioms (i)–(iii) above and conclude that $w_i'(\mathcal{E}) = w_i(\mathcal{E})$.

5. Determine the cohomology ring $H^*(BGL_n(\mathbb{R}))$.