Topological Hochschild Homology and the de Rham-Witt complex

Lars Hesselholt

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$$\Delta \xrightarrow{\iota} \Lambda$$

simpl. cat. cyclic cat.

$$\mathrm{ob}\,\Delta = \mathrm{ob}\,\Lambda$$

$$\mathrm{Aut}_{\Lambda}([m]) = \left\{ \begin{array}{l} \text{cyclic perm. of} \\ [m] = \{0, 1, \ldots, m\} \end{array} \right\}$$

$$\mathrm{Hom}_{\Delta}([m],[n]) \times \mathrm{Aut}_{\Lambda}([m]) \xrightarrow[\sim]{\circ} \mathrm{Hom}_{\Lambda}([m],[n]).$$

Composition

$$\begin{tikzcd}
{[k]} \arrow[r, "u"] \arrow[dr] & {[k]} \arrow[r, "f"] \arrow[d, "f^* v"] & {[m]} \arrow[d, "v"] \\
& {[k]} \arrow[r, "v_* f"] \arrow[dr] & {[m]} \arrow[d, "g"] \\
& & {[n]}
\end{tikzcd}$$

$$\begin{tikzcd}
{[k]} = f^{-1}(v^{-1}(0)) \amalg \cdots \amalg f^{-1}(v^{-1}(m)) \arrow[d, "f^* v"'] \\
{[k]} = f^{-1}(0) \amalg \cdots \amalg f^{-1}(m)
\end{tikzcd}$$

$$v_* f = v \circ f \circ (f^* v)^{-1} . \qquad //$$

$$I \xrightarrow{f} J \quad \rightsquigarrow \quad \mathcal{E}^J \xrightarrow{f_*} \mathcal{E}^I$$

$$X \longmapsto X \circ f$$

$$f^* \dashv f_* \dashv f^!$$

left Kan extension $\qquad$ right Kan extension

$$(f^* X)(j) = \operatorname*{colim}_{f/j} X(i)$$

$$(f^! X)(j) = \lim_{f \backslash j} X(i)$$

ob f/j : $\quad i \quad f(i) \xrightarrow{v} j$

mor f/j :

$$\begin{tikzcd} i \arrow[d, "u"] & f(i) \arrow[r, "v"] \arrow[d, "f(u)"] & j \arrow[d, equal] \\ i' & f(i') \arrow[r, "v'"] & j \end{tikzcd}$$

ob $f \backslash j$: $\quad i \quad j \xrightarrow{v} f(i)$

mor $f \backslash j$:

$$\begin{tikzcd} i \arrow[d, "u"] & j \arrow[r, "v"] \arrow[d, equal] & f(i) \arrow[d, "f(u)"] \\ i' & j \arrow[r, "v'"] & f(i') \end{tikzcd}$$

$$\mathrm{Sets}^{\Delta^{op}} \xrightarrow{\iota^*} \mathrm{Sets}^{\Lambda^{op}}$$

$$Y \longmapsto \text{“free cyclic set generated by } Y\text{”}$$

$$\mathrm{Hom}_{\Lambda}([m],[n]) = \mathrm{Hom}_{\Lambda^{op}}([n],[m])$$

$$[m] \xrightarrow{\theta = f \circ u} [n] \qquad [n] \xrightarrow{\theta^{op} = u^{op} \circ f^{op}} [m]$$

Prop Let Y be a simplicial set. Then there is a canonical isom.

$$\mathrm{Aut}_{\Delta^{op}}([n]) \times Y_n \xrightarrow{\sim} (\iota^* Y)_n$$

and if $\theta = \rho \circ u : [m] \to [n]$ then

$$\begin{tikzcd}
(\iota^* Y)_n \arrow[d, "\theta^*"] & \mathrm{Aut}_{\Delta^{op}}([n]) \times Y_n \arrow[l, "\sim"'] \arrow[d] & \ni & (v^{op}, y) \arrow[d, mapsto] \\
(\iota^* Y)_m & \mathrm{Aut}_{\Delta^{op}}([m]) \times Y_m \arrow[l, "\sim"'] & \ni & ((\rho^* v \circ u)^{op}, (v_* \rho)^* y)
\end{tikzcd}$$

Pf The inclusion

$$\mathrm{Aut}_{\Delta^{op}}([n]) \hookrightarrow \iota / [n]$$

$$v^{op} \longmapsto ([n], [n] \xrightarrow{v^{op}} [n])$$

is cofinal:

$$\begin{tikzcd}
{[m]} \arrow[d, "g^{op}"] & {[m]} \arrow[r, "v^{op} \circ g^{op}"] \arrow[d, "g^{op}"] & {[n]} \arrow[d, equal] \\
{[n]} & {[n]} \arrow[r, "v^{op}"] & {[n]}
\end{tikzcd}$$

so induces an isom.

$$\coprod_{\mathrm{Aut}_{\Delta^{op}}([n])} Y_n = \operatorname*{colim}_{\mathrm{Aut}_{\Delta^{op}}([n])} Y_n \xrightarrow{\sim} \operatorname*{colim}_{\iota/[n]} Y_m .$$

The inverse maps:

$$([n] \xrightarrow{v^{op}} [n];\ g^* y) \longleftarrow ([m], [m] \xrightarrow{v^{op} \circ g^{op}} [n];\ y) .$$

Calculate Θ^*:

$$\begin{array}{ccc} \mathrm{Aut}_{\Delta^{op}}([n]) & \hookrightarrow \iota/[n] & \xrightarrow{\Theta^*} \iota/[m] \\ [n] \xrightarrow{v^{op}} [n] & \longmapsto & ([n], [n] \xrightarrow{v^{op}} [n] \xrightarrow{\theta^{op}} [m]) \\ & & \parallel \\ & & ([n], [n] \xrightarrow{(v_* f)^{op}} [m] \xrightarrow{(f^* v \circ u)^{op}} [m]) \end{array}$$

$$\begin{tikzcd} {[m]} & {[m]} \arrow[l, "u^{op}"'] & {[n]} \arrow[l, "f^{op}"'] \\ & {[m]} \arrow[u, "(f^* v)^{op}"] & {[n]} \arrow[l, "(v_* f)^{op}"] \arrow[u, "v^{op}"'] \end{tikzcd} .$$

//

<u>Cor</u> There is a canonical homeomorphism

$$\Delta^1/\partial\Delta^1 \xrightarrow{\sim} |\iota_* \iota^*(pt)| .$$

<u>pf</u> By prop.,

$$\begin{tikzcd} \iota_*\iota^*(pt)_n \arrow[d, "f^*"] & \mathrm{Aut}_{\Lambda^{op}}([n]) \arrow[l, "\sim"'] \arrow[d] & \ni & v \arrow[d, mapsto] \\ \iota_*\iota^*(pt)_m & \mathrm{Aut}_{\Lambda^{op}}([m]) \arrow[l, "\sim"'] & \ni & f^* v \end{tikzcd}$$

so $\iota_*\iota^*(pt)$ has one 0-simpl., one non-deg. 1-simpl., and every n-simpl. for $n \geq 2$ is degenerate. //

The topological standard simplices

$$\Delta^n = |\Delta[n]|$$

form a cocyclic space. Indeed,

$$\Delta[n] := \mathrm{Hom}_{\Delta}(-,[n]) = \mathrm{Hom}_{\Delta}(-,\iota_*[n])$$

$$= \mathrm{Hom}_{\Lambda}(\iota^*(-),[n]) .$$

Prop There is a canonical (non-simpl.) homeomorphism

$$|L_* L^* Y| \xrightarrow[\sim]{\ell} |L_* L^*(pt) \times Y|$$

pf The map ℓ is given by

$$\ell(v^{op}, y; z) = ((v^{-1})^{op}, y; v_* z).$$

Clearly ℓ is a homeo. if it is well-defined. Let $f \in \mathrm{Hom}_\Delta([m],[n])$; then

$$\ell(v^{op}, y; f_* z) = ((v^{-1})^{op}, y; v_*(f_* z))$$

$$= ((v^{-1})^{op}, y; (v_* f)_*((f^* v)_* z))$$

$$\begin{tikzcd} {[m]} \arrow[r, "f"] \arrow[d, "f^* v"'] & {[n]} \arrow[d, "v"] \\ {[m]} \arrow[r, "v_* f"] & {[n]} \end{tikzcd}$$

$$= ((v_* f)^*((v^{-1})^{op}, y); (f^* v)_* z)$$

$$\overset{\text{prop}}{=} (((v_* f)^*(v^{-1}))^{op}, (v_* f)^* y; (f^* v)_* z)$$

$$= \left(((f^* v)^{-1})^{op}, (v_* f)^* y ; (f^* v)_* z \right)$$

$$\begin{tikzcd}
{[m]} \arrow[r, "f"] \arrow[d, "f^* v"] & {[n]} \arrow[d, "v"] \\
{[m]} \arrow[r, "v_* f"] \arrow[d, "(v_* f)^* (v^{-1})"] & {[n]} \arrow[d, "v^{-1}"] \\
{[m]} \arrow[r, "(v^{-1})_* v_* f"] & {[n]}
\end{tikzcd}$$

$$= l \left((f^* v)^{op}, (v_* f)^* y ; z \right)$$

$$\overset{\text{prop}}{=} l \left(f^* (v^{op}, y) ; z \right) .$$

//

<u>Cor</u> There is a canonical (non-simpl.) homeomorphism

$$|\iota_* \iota^* Y| \xrightarrow[\sim]{h} |\iota_* \iota^* (pt)| \times |Y| .$$

<u>pf</u> The map h is the composite of l and $|\iota_* \iota^* (pt) \times Y| \xrightarrow{\sim} |\iota_* \iota^* (pt)| \times |Y|$. //

Let X be a cyclic set and define

$$|L_* L^* L_*(pt)| \times |L_* X| \xrightarrow{\mu_X} |L_* X|$$

to be the composite

$$|L_* L^* L_*(pt)| \times |L_* X| \underset{\sim}{\xrightarrow{h_{pt} \times id}} |L_* L^* L_*(pt)| \times |L_* X|$$

$$\underset{\sim}{\xleftarrow{h_X}} |L_* L^* L_* X| \xrightarrow{|L_* \varepsilon|} |L_* X|$$

where $\varepsilon : L^* L_* X \to X$ is the counit of the adjunction (L^*, L_*).

Prop (i) The space $\mathbb{T} = |L_* L^* L_*(pt)|$ is a group with multiplication μ_{pt} and inverse h_{pt}.

(ii) If X is a cyclic set, then

$$\mathbb{T} \times |L_* X| \xrightarrow{\mu_X} |L_* X|$$

is an action of $\mathbb{T}$ on $|L_* X|$.

Pf There are two non-deg. simpl. in

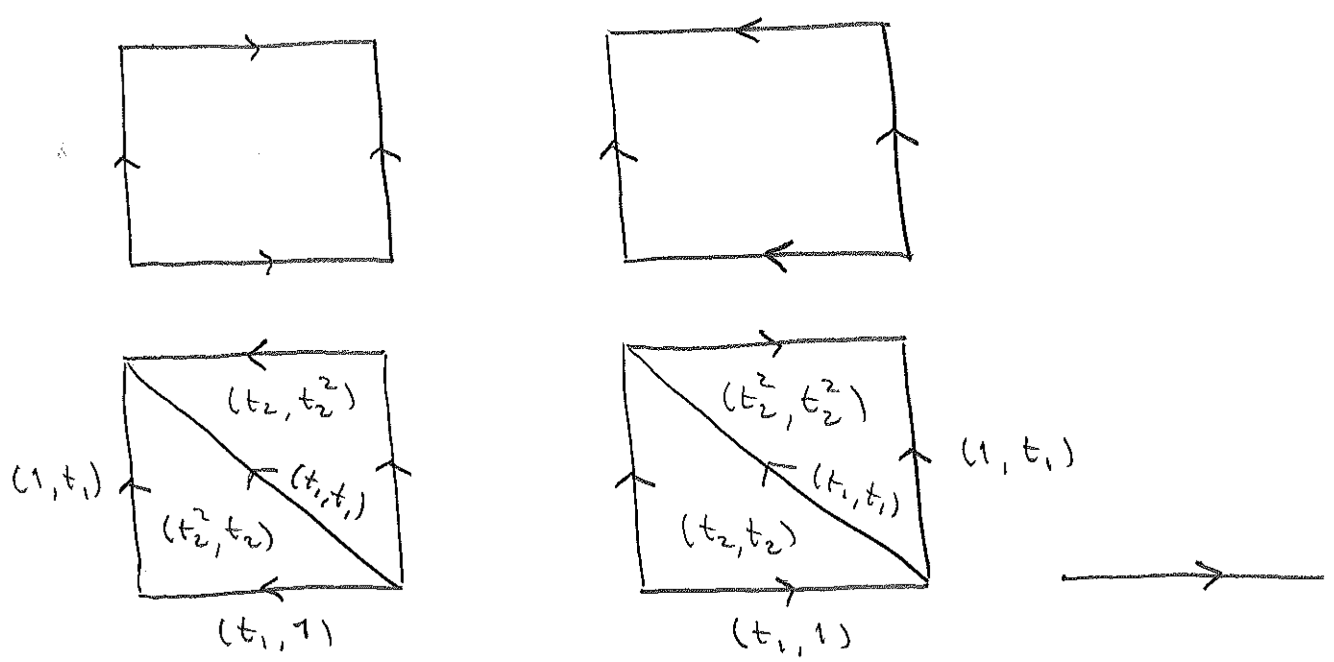
$$L_* L^* L_* L^* L_*(pt)_2 \xleftarrow{\sim} \mathrm{Aut}_{\Lambda^{op}}([2]) \times \mathrm{Aut}_{\Lambda^{op}}([2])$$

They are (t_2, t_2) and (t_2^2, t_2^2), where $t_2 \in \mathrm{Aut}_{\Delta^{op}}([2])$ is a generator. Similarly, there are two non-deg. simpl. in

$$(\iota_*\iota^*\iota_*(pt) \times \iota_*\iota^*\iota_*(pt))_2 \xleftarrow{\sim} \mathrm{Aut}_{\Delta^{op}}([2]) \times \mathrm{Aut}_{\Delta^{op}}([2]).$$

They are (t_2, t_2^2) and (t_2^2, t_2). Let also $t_1 \in \mathrm{Aut}_{\Delta^{op}}([1])$ be the generator. Then the map μ_{pt} is given by the following cartoon, which is made such that the various homeo. preserves the position of points in the squares.

$$\mathbb{T} \times \mathbb{T} \xrightarrow[\sim]{h \times id} \mathbb{T} \times \mathbb{T} \xleftarrow[\sim]{pr_1 \times pr_2}$$



$$|\iota_*\iota^*\iota_*(pt) \times \iota_*\iota^*\iota_*(pt)| \xleftarrow[\sim]{h_{\iota_*\iota^*\iota_*(pt)}} |\iota_*\iota^*\iota_*\iota^*\iota_*(pt)| \xrightarrow{\varepsilon} \mathbb{T}$$

//

We give generators and relations for the cyclic (index) category Λ :

Generators :

$$d^i : [n-1] \longrightarrow [n] \qquad 0 \leq i \leq n$$

$$s^i : [n+1] \longrightarrow [n] \qquad 0 \leq i \leq n$$

$$\tau_n : [n] \longrightarrow [n]$$

given by

$$d^i(j) = \begin{cases} j & 0 \leq j < i \\ j+1 & i \leq j \leq n-1 \end{cases}$$

$$s^i(j) = \begin{cases} j & 0 \leq j \leq i \\ j-1 & i < j \leq n+1 \end{cases}$$

$$\tau_n(j) = \begin{cases} n & i = 0 \\ j-1 & 0 < j \leq n \end{cases}.$$

Relations

$$d^j d^i = d^i d^{j-1} \qquad i < j$$

$$s^j s^i = s^i s^{j+1} \qquad i \leq j$$

$$s^j d^i = \begin{cases} d^i s^{j-1} & i < j \\ id & i = j \text{ or } i = j+1 \\ d^{i-1} s^j & i > j+1 \end{cases}$$

$$\tau_n d^i = \begin{cases} d^n & i = 0 \\ d^{i-1} \tau_{n-1} & 1 \le i \le n \end{cases}$$

$$\tau_n s^i = \begin{cases} s^n \tau_{n+1}^2 & i = 0 \\ s^{i-1} \tau_{n+1} & 1 \le i \le n \end{cases}.$$

$$\tau_n^{n+1} = id.$$

Let d_i, s_i, and t_n be the opposite of d^i, s^i, and τ_n. Then Δ^{op} has

Generators:

$d_i : [n] \longrightarrow [n-1] \qquad 0 \le i \le n$

$s_i : [n] \longrightarrow [n+1] \qquad 0 \le i \le n$

$t_n : [n] \longrightarrow [n]$

Relations:

$$d_i d_j = d_{j-1} d_i \qquad i < j$$

$$s_i s_j = s_{j+1} s_i \qquad i \leq j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & i < j \\ \mathrm{id} & i = j \text{ or } i = j+1 \\ s_j d_{i-1} & i > j+1 \end{cases}$$

$$d_i t_n = \begin{cases} d_n & i = 0 \\ t_{n-1} d_{i-1} & 0 < i \leq n \end{cases}$$

$$s_i t_n = \begin{cases} t_{n+1}^2 s_n & i = 0 \\ t_{n+1} s_{i-1} & 0 < i \leq n \end{cases}$$

$$t_n^{n+1} = \mathrm{id}.$$

We recall the adjunction

$$\mathrm{Sets}^{\Delta^{op}} \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} \mathrm{Sets}^{\Lambda^{op}}$$

Prop Let Y be a simpl. set. Then there is a natural isomorphism

$$\mathrm{Aut}_{\Lambda^{op}}([n]) \times Y_n \xrightarrow{\sim} (i^* Y)_n$$

with cyclic structure maps

$$d_i(t_n^s, y) = \begin{cases} (t_{n-1}^{s-1}, d_{n+1+i-s}(y)) & 0 \leq i < s \\ (t_{n-1}^{s}, d_{i-s}(y)) & s \leq i \leq n \end{cases}$$

$$s_i(t_n^s, y) = \begin{cases} (t_{n+1}^{s+1}, s_{n+1+i-s}(y)) & 0 \leq i < s \\ (t_{n+1}^{s}, s_{i-s}(y)) & s \leq i \leq n \end{cases}$$

$$t_n(t_n^s, y) = (t_n^{s+1}, y).$$

Proof From last time, we have

$$d_i(t_n^s, y) = ((d^{i*}\tau_n^s)^{op}, (\tau_{n*}^s d^i)^* y)$$

$$s_i(t_n^s, y) = ((s^{i*}\tau_n^s)^{op}, (\tau_{n*}^s s^i)^* y)$$

$$t_n(t_n^s, y) = ((\tau_n^s \circ \tau_n)^{op}, y).$$

We recall that if $f \in \mathrm{Hom}_\Delta([m],[n])$ and $u \in \mathrm{Aut}_\Delta([n])$, then the maps $f^*u \in \mathrm{Aut}_\Delta([m])$ and $u_*f \in \mathrm{Hom}_\Delta([m],[n])$ are uniquely determined by the requirements that

$$\begin{tikzcd}
{[m]} \arrow[r, "f"] \arrow[d, "f^*u"] & {[n]} \arrow[d, "u"] \\
{[m]} \arrow[r, "u_*f"] & {[n]}
\end{tikzcd}$$

be a commutative diagr. of maps of sets and that the restriction of f^*u to $f^{-1}(i)$ be order-preserving for all $0 \le i \le n$. Hence, the statement for the face maps follows from the commutativity of the diagrams

$0 \le i < s$:

$$\begin{tikzcd}
{[n-1]} \arrow[r, "d^i"] \arrow[d, "\tau_{n-1}^{s-1}"] & {[n]} \arrow[d, "\tau_n^s"] \\
{[n-1]} \arrow[r, "d^{n+1+i-s}"] & {[n]}
\end{tikzcd}$$

$s \le i \le n$:

$$\begin{array}{ccc} [n-1] & \xrightarrow{d^i} & [n] \\ \big\downarrow \tau^s_{n-1} & & \big\downarrow \tau^s_n \\ [n-1] & \xrightarrow{d^{i-s}} & [n] \end{array}$$

In the case of the degeneracy maps we must also check the second requirement:

$0 \le i < s$:

$$\begin{array}{ccc} [n+1] & \xrightarrow{s^i} & [n] \\ \big\downarrow \tau^{s+1}_{n+1} & & \big\downarrow \tau^s_n \\ [n+1] & \xrightarrow{s^{n+1+i-s}} & [n] \end{array}$$

$$(s^i)^{-1}(i) = \{i, i+1\}$$

$$\tau^{s+1}_{n+1}(i) = n+2+i-(s+1) < n+2+i+1-(s+1) = \tau^{s+1}_{n+1}(i+1).$$

$s \le i \le n$:

$$\begin{tikzcd} {[n+1]} \arrow[r, "s^i"] \arrow[d, "\tau^s_{n+1}"] & {[n]} \arrow[d, "\tau^s_n"] \\ {[n+1]} \arrow[r, "s^{i-s}"] & {[n]} \end{tikzcd}$$

$$(s^i)^{-1}(i) = \{i, i+1\}$$

$$\tau^s_{n+1}(i) = i-s \le i+1-s = \tau^s_{n+1}(i+1) . \quad //$$

Let us check a couple of relations :

$$s_0 t_n (t^s_n, y) = s_0 (t^{s+1}_n, y)$$

$$= (t^{s+2}_{n+1}, s_{n+1-(s+1)}(y))$$

$$t^2_{n+1} s_n (t^s_n, y) = (t^{s+2}_{n+1}, s_{n-s}(y)) \quad \checkmark$$

$$d_0 t_n (t^s_n, y) = d_0 (t^{s+1}_n, y)$$

$$= \begin{cases} (t^s_{n-1}, d_{n+1-(s+1)}(y)) & 0 \le s < n \\ (t^0_{n-1}, d_0(y)) & s = n \end{cases}$$

$$= (t^s_{n-1}, d_{n-s}(y)) = d_n (t^s_n, y) . \quad \checkmark$$

The prop. holds more generally if the cat. of sets is replaced by any category that has all finite colimits. It gives an iso.

$$\coprod_{\mathrm{Aut}_{\Lambda^{op}}([n])} Y_n \xrightarrow{\sim} (\iota^* Y)_n .$$

We consider the category of k-modules, where k is a comm. ring. If A is a simpl. k-module, we write A_* for the associated chain cx.

<u>Lemma</u> Let A be a cyclic k-module. Then there is a natural isom. of chain complexes of k-modules

$$(\iota_* \iota^* A)_* \underset{\sim}{\xrightarrow{\ell}} (\iota_* \iota^* (k) \otimes A)_*$$

$$v^{op} \otimes a \longmapsto \mathrm{sgn}(v) . (v^{-1})^{op} \otimes a .$$

<u>pf</u> We check that the following diagr. commutes.

$$\begin{array}{ccc} (\iota_* \iota^* A)_n & \xrightarrow{\ell} & (\iota_* \iota^* (k) \otimes A)_n \\ \downarrow d & & \downarrow d \\ (\iota_* \iota^* A)_{n-1} & \xrightarrow{\ell} & (\iota_* \iota^* (k) \otimes A)_{n-1} \end{array}$$

$$d(\ell(t_n^s \otimes a)) = \sum_{i=0}^{n} (-1)^i d_i(\ell(t_n^s \otimes a))$$

$$= \sum_{i=0}^{n} (-1)^{i+ns} d_i(t_n^{n+1-s}) \otimes d_i a$$

$$= \sum_{i=0}^{n-s} (-1)^{i+ns} t_{n-1}^{n-s} \otimes d_i a$$

$$+ \sum_{i=n+1-s}^{n} (-1)^{i+ns} t_{n-1}^{n+1-s} \otimes d_i a .$$

$$\ell(d(t_n^s \otimes a)) = \ell\left(\sum_{j=0}^{n} (-1)^j d_j(t_n^s \otimes a)\right)$$

$$= \sum_{j=0}^{s-1} (-1)^j \ell(t_{n-1}^{s-1} \otimes d_{n+1+j-s} a)$$

$$+ \sum_{j=s}^{n} (-1)^j \ell(t_{n-1}^{s} \otimes d_{j-s} a)$$

$$= \sum_{j=0}^{s-1} (-1)^{j+(n-1)(s-1)} t_{n-1}^{n-(s-1)} \otimes d_{n+1+j-s} a$$

$$+ \sum_{j=s}^{n} (-1)^{j+(n-1)s} t_{n-1}^{n-s} \otimes d_{j-s} a$$

$$= \sum_{i=n+1-s}^{n} (-1)^{i+ns}\, t_{n-1}^{n+1-s} \otimes d_i a \qquad (i = n+1+j-s)$$

$$+ \sum_{i=0}^{n-s} (-1)^{i+ns}\, t_{n-1}^{n-s} \otimes d_i a \qquad (i = j-s).$$

So l is a chain map. It is clearly an isomorphism. //

We recall that if A and A' are simpl. k-modules, then the Eilenberg-MacLane map is a quasi-isom.

$$A_* \otimes A'_* \xrightarrow[\sim]{\ \theta\ } (A \otimes A')_*$$

that to $a \otimes a' \in A_p \otimes A'_q$ assigns

$$\sum_{\sigma \in \Sigma_{p+q}} \operatorname{sgn}(\sigma)\, s_{\sigma(p+q)-1} \cdots s_{\sigma(p+1)-1}\, a \otimes s_{\sigma(p)-1} \cdots s_{\sigma(1)-1}\, a'$$

in $(A \otimes A')_{p+q} = A_{p+q} \otimes A'_{p+q}$. The sum runs over all (p,q)-shuffles.

If A is a cyclic k-module, we have the counit

$$\iota^* \iota_* A \xrightarrow{\ \varepsilon\ } A$$

$$v^{op} \otimes a \longmapsto v^* a .$$

Let A be a cyclic k-module. We consider the chain map

$$(L_* L^* L_*(k))_* \otimes (L_* A)_* \xrightarrow{\mu_A} (L_* A)_*$$

defined as the composite

$$(L_* L^* L_*(k))_* \otimes (L_* A)_* \xrightarrow[\sim]{1 \otimes \mathrm{id}}$$

$$(L_* L^* L_*(k))_* \otimes (L_* A)_* \xrightarrow{\Theta}$$

$$(L_* L^* L_*(k) \otimes A)_* \xleftarrow[\sim]{l}$$

$$(L_* L^* L_* A)_* \xrightarrow{L_* \varepsilon} (L_* A)_* .$$

This is analogous to the $\mathbb{T}$-action on the realization of a cyclic set. Indeed, as cyclic k-modules

$$L^* L_*(k) \xrightarrow{\sim} k\{L^* L_*(pt)\} .$$

Let A be a (not necessarily comm.) unital k-algebra. The Hochschild complex of A/k is the cyclic k-module $HH(A/k)$ with n-simpl.

$$HH(A/k)_n = A \otimes_k \cdots \otimes_k A \qquad (n+1 \text{ factors})$$

and cyclic structure maps

$$d_i(a_0 \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \qquad 0 \le i < n$$

$$= a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} \qquad i = n$$

$$s_i(a_0 \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n \qquad 0 \le i \le n$$

$$t_n(a_0 \otimes \cdots \otimes a_n) = a_n \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}$$

We compute Connes' operator

$$HH(A/k)_n \xrightarrow{B} HH(A/k)_{n+1}$$

$$a_0 \otimes \cdots \otimes a_n \longmapsto \mu(t_1 \otimes (a_0 \otimes \cdots \otimes a_n))$$

<u>Lemma</u> Connes' operator is given on the chain level by the formula

$$B(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n} (-1)^{ni}\, 1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1}.$$

<u>pf</u> We follow the def. of μ.

$$t_1 \otimes (a_0 \otimes \cdots \otimes a_n) \overset{\text{lord}}{\longmapsto} -t_1 \otimes (a_0 \otimes \cdots \otimes a_n)$$

$$\overset{\theta}{\longmapsto} -\sum_{i=0}^{n} (-1)^i\, s_n \cdots \hat{s}_i \cdots s_0 t_1 \otimes s_i (a_0 \otimes \cdots \otimes a_n)$$

$$= -\sum_{i=0}^{n}(-1)^{i}\, t_{n+1}^{i+1} \otimes (a_0\otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1}\otimes \cdots \otimes a_n)$$

$$\overset{\ell}{\longleftarrow\!\!\shortmid} -\sum_{i=0}^{n}(-1)^{i+(n+2)(n+1-i)}\, t_{n+1}^{n+1-i} \otimes (a_0\otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1}\otimes \cdots \otimes a_n)$$

$$\overset{\varepsilon}{\longmapsto} -\sum_{i=0}^{n}(-1)^{i+(n+2)(n+1-i)}\, 1\otimes a_{i+1}\otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_i$$

$$= \sum_{j=0}^{n}(-1)^{nj}\, 1\otimes a_j \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{j-1}\,. \qquad //$$

2/12/04

$$\begin{tikzcd} \Delta \arrow[r, "sd^r"] & \Delta \end{tikzcd}$$

$$\begin{tikzcd} {[n-1]} \arrow[r, mapsto] \arrow[d, "f"] & {[rn-1]} \arrow[d, "sd^r(f)"] & \ni & an+b \arrow[d, mapsto] \\ {[m-1]} \arrow[r, mapsto] & {[rm-1]} & \ni & am+f(b) \end{tikzcd}$$

$0 \leq a < r$

$0 \leq b < n$

$$\begin{tikzcd} \Delta^{op} \arrow[r, "sd^r"] \arrow[rr, bend right, "sd_r Y"'] & \Delta^{op} \arrow[r, "Y"] & \mathcal{C} \end{tikzcd}$$

r-fold edge-wise subdivision

$$(sd_r Y)_n = Y_{r(n+1)-1}$$

$$(sd_r Y)_n \xrightarrow{d_i'} (sd_r Y)_{n-1} \qquad 0 \leq i \leq n$$

$$(sd_r Y)_n \xrightarrow{s_i'} (sd_r Y)_{n+1} \qquad 0 \leq i \leq n$$

$$d_i' = d_i \circ d_{i+(n+1)} \circ \cdots \circ d_{i+(r-1)(n+1)}$$

$$s_i' = s_{i+(r-1)(n+2)} \circ \cdots \circ s_{i+(n+2)} \circ s_i$$

Ex Calc. $sd_r \Delta[1]$. An n-simplex is a morphism in Δ

$$[r(n+1)-1] =: \underbrace{[n] \amalg \cdots \amalg [n]}_{r} \longrightarrow [1].$$

Let $\varepsilon : [n] \to [1]$ denote the constant map with value ε. There are $r+1$ 0-simpl.:

$$\underbrace{0 \amalg \cdots \amalg 0}_{i} \amalg \underbrace{1 \amalg \cdots \amalg 1}_{r-i} \qquad 0 \le i \le r$$

and $2r+1$ 1-simpl.

$$\underbrace{0 \amalg \cdots \amalg 0}_{i} \amalg \underbrace{1 \amalg \cdots \amalg 1}_{r-i} \qquad 0 \le i \le r$$

$$\underbrace{0 \amalg \cdots \amalg 0}_{i} \amalg \mathrm{id} \amalg \underbrace{1 \amalg \cdots \amalg 1}_{r-1-i} \qquad 0 \le i \le r-1$$

of which the first $r+1$ are degenerate:

$$s_0(\underbrace{0 \amalg \cdots \amalg 0}_{i} \amalg \underbrace{1 \amalg \cdots \amalg 1}_{r-i}) = \underbrace{0 \amalg \cdots \amalg 0}_{i} \amalg \underbrace{1 \amalg \cdots \amalg 1}_{r-i}.$$

Similarly, all n-simplices, for $n \ge 2$, are degenerate. Calc. face maps:

$$d_0'(\underbrace{0 \amalg \cdots \amalg 0}_{i} \amalg \mathrm{id} \amalg \underbrace{1 \amalg \cdots \amalg 1}_{r-1-i}) = \underbrace{0 \amalg \cdots \amalg 0}_{i} \amalg \underbrace{1 \amalg \cdots \amalg 1}_{r-i}$$

$$d_1'(\underbrace{0 \amalg \cdots \amalg 0}_{i} \amalg \mathrm{id} \amalg \underbrace{1 \amalg \cdots \amalg 1}_{r-1-i}) = \underbrace{0 \amalg \cdots \amalg 0}_{i+1} \amalg \underbrace{1 \amalg \cdots \amalg 1}_{r-1-i}$$

So there is an affine homeomorphism

$$[0,r] \xrightarrow{\sim} |\, sd_r \Delta[1] \,|$$

that takes the integer $i \in [0,r]$ to the vertex $\underbrace{0 \amalg \cdots \amalg 0}_{i} \amalg \underbrace{1 \amalg \cdots \amalg 1}_{r-i}$. //

Let $\Delta^{n-1} \subset \mathbb{R}^n$ be the convex hull of the standard basis and define

$$\Delta^{n-1} \xrightarrow{d_r} \Delta^{rn-1}$$

$$z \longmapsto \frac{z}{r} \oplus \cdots \oplus \frac{z}{r}$$

We define

$$|\, sd_r Y \,| \xrightarrow{D_r} |\, Y \,|$$

to be the natural (non-simpl.) map induced from the maps

$$Y_{rn-1} \times \Delta^{n-1} \xrightarrow{id \times d^r} Y_{rn-1} \times \Delta^{rn-1}.$$

Prop The map

$$|\, sd_r Y \,| \xrightarrow{D_r} |\, Y \,|$$

is a homeomorphism.

pf $Y = \Delta[1]$: Let

$$f_i = \underbrace{0 \amalg \cdots \amalg 0}_{i} \amalg \mathrm{id} \amalg \underbrace{1 \amalg \cdots \amalg 1}_{r-1-i} : [2r-1] \to [1]$$

be the i'th non-deg. 1-simplex of $sd_r \Delta[1]$. As a $(2r-1)$-simplex of $\Delta[1]$, f_i is degenerate : $f_i = f_i^*(\mathrm{id}_{[1]})$. So

$$D_r(f_i, z) = (f_i, \tfrac{z}{r} \oplus \cdots \oplus \tfrac{z}{r})$$

$$= (\mathrm{id}_{[1]}, f_{i*}(\tfrac{z}{r} \oplus \cdots \oplus \tfrac{z}{r})).$$

We have

$$\Delta^{2r-1} \xrightarrow{f_{i*}} \Delta^1$$

$$(x_1, \ldots, x_{2r}) \longmapsto (x_1 + \cdots + x_{2i+1}, x_{2i+2} + \cdots + x_{2r}).$$

So if $z = (x_1, x_2) \in \Delta^1$, then

$$f_{i*}(d_r(z)) = (\tfrac{i}{r} + \tfrac{x_1}{r}, \tfrac{r-1-i}{r} + \tfrac{x_2}{r}).$$

Hence, D_r is a homeo. for $Y = \Delta[1]$.

$Y = \Delta[n]$: There is a retraction

$$\Delta[n] \xhookrightarrow{\iota} (\Delta[1])^n \xrightarrow{\rho} \Delta[n]$$

defined as follows. The map ι is

given by the n non-constant maps $[n] \to [1]$, and the map ρ is given by addition: If $f_1, \dots, f_n : [m] \to [1]$ are order-preserving, then so is the map

$$\begin{aligned} [m] &\xrightarrow{\rho(f_1,\dots,f_n)} [n] \\ j &\longmapsto f_1(j) + \dots + f_n(j). \end{aligned}$$

Consider

$$\begin{tikzcd}
{|sd_r \Delta[n]|} \arrow[r, "D_r"] \arrow[d, "\iota"] & {|\Delta[n]|} \arrow[d, "\iota"] \\
{|sd_r (\Delta[1])^n|} \arrow[r, "D_r"] \arrow[d, "\rho"] & {|(\Delta[1])^n|} \arrow[d, "\rho"] \\
{|sd_r \Delta[n]|} \arrow[r, "D_r"] & {|\Delta[n]|}
\end{tikzcd}$$

The middle D_r is a homeo. since

$$\begin{tikzcd}
{|sd_r (\Delta[1])^n|} \arrow[r, "D_r"] \arrow[d, "\sim"] & {|(\Delta[1])^n|} \arrow[d, "\sim"] \\
{|sd_r \Delta[1]|^n} \arrow[r, "D_r^n", "\sim"'] & {|\Delta[1]|^n}
\end{tikzcd}$$

It follows that

$$|sd_r \Delta[n]| \xrightarrow{D_r} |\Delta[n]|$$

is a continuous bijection and hence a homeomorphism.

General Y: Use the co-equalizer

$$\coprod_{[m]\to[n]} Y_n \times \Delta[m] \rightrightarrows \coprod_{[n]} Y_n \times \Delta[n] \to Y \; .$$

Since sd_r and $|-|$ preserves colimits, the general case follows. //

<u>Rem</u> Note that

$$sd_{rs} Y = sd_r \, sd_s Y$$

and that the diagr.

$$\begin{tikzcd} {|sd_{rs} Y|} \arrow[rr, "D_r", "\sim"'] \arrow[dr, "D_{rs}", "\sim"'] & & {|sd_s Y|} \arrow[dl, "D_s", "\sim"'] \\ & {|Y|} & \end{tikzcd}$$

commutes. //

Subdivision of cyclic objects. Construct a commutative (pull-back) diagram

$$\begin{tikzcd} \Delta \arrow[r, "sd^r"] \arrow[d, "\iota_r"] & \Delta \arrow[d, "\iota"] \\ \Lambda_r \arrow[r, "sd^r"] & \Lambda \end{tikzcd}.$$

Define

$$\operatorname{ob} \Lambda_r = \operatorname{ob} \Delta$$

$$\operatorname{Aut}_{\Lambda_r}([n]) = \operatorname{Aut}_{\Lambda}(sd^r([n]))$$

$$\operatorname{Hom}_{\Delta}([m],[n]) \times \operatorname{Aut}_{\Lambda_r}([m])$$

$$\xrightarrow[\sim]{\circ} \operatorname{Hom}_{\Lambda_r}([m],[n]).$$

Composition:

$$\begin{tikzcd} {[k]} \arrow[r, "u"] & {[k]} \arrow[r, "f"] \arrow[d, "f^{\#}y"] & {[m]} \arrow[d, "y"] \\ & {[k]} \arrow[r, "y_{\#}f"] & {[m]} \arrow[d, "g"] \\ & & {[n]} \end{tikzcd}$$

with $f^{\#}v$ and $v_{\#}f$ determined by

$$f^{\#}v = sd^r(f)^* v$$

$$sd^r(v_{\#}f) = v_* sd^r(f).$$

This makes the functor

$$\Lambda_r \xrightarrow{sd^r} \Lambda$$

$$[n] \longmapsto [r(n+1)-1]$$

$$(u, f) \longmapsto (u, sd^r(f))$$

well-defined. The category Λ_r is generated by the morphisms

$$d^i : [n-1] \longrightarrow [n] \qquad 0 \leq i \leq n$$

$$s^i : [n+1] \longrightarrow [n] \qquad 0 \leq i \leq n$$

$$\tau_{r,n} : [n] \longrightarrow [n]$$

with d^i and s^i defined as before and with

$$\begin{array}{ccc} \mathrm{Aut}_{\Lambda_r}([n]) & = & \mathrm{Aut}_{\Lambda}([r(n+1)-1]) \\ \cup & & \cup \\ \tau_{r,n} & = & \tau_{r(n+1)-1} \end{array}$$

subject to the same relations as before except for the last one:

$$\tau_{r,n}^{r(n+1)} = \mathrm{id}.$$

Let d_i, s_i, and $t_{r,n}$ be the generators of Δ_r^{op} corresponding to d^i, s^i, and $\tau_{r,n}$.

<u>Lemma</u> Let X be a cyclic object. Then the maps

$$(sd_r X)_n \xrightarrow{t_{r,n}^{n+1}} (sd_r X)_n$$

define an automorphism of the simpl. object $i_{r*}\, sd_r X$ of order r.

Pf Recall that

$$d_i t_{r,n} = \begin{cases} d_n & 0 = i \\ t_{r,n-1} d_{i-1} & 0 < i \le n. \end{cases}$$

So

$$d_i t_{r,n}^{n+1} = t_{r,n-1}^{i} d_0 t_{r,n}^{n+1-i}$$

$$= t_{r,n-1}^{i} d_n t_{r,n}^{n-i} = t_{r,n-1}^{n} d_i .$$

Similarly, $s_i t_{r,n}^{n+1} = t_{r,n+1}^{n+2} s_i$. //

Consider the adjunction

$$\mathrm{Sets}^{\Delta^{op}} \underset{\iota_{r*}}{\overset{\iota_r^*}{\rightleftarrows}} \mathrm{Sets}^{\Lambda_r^{op}}.$$

Prop Let Y be a simplicial set. Then there is a canonical isom.

$$\mathrm{Aut}_{\Lambda_r^{op}}([n]) \times Y_n \xrightarrow{\sim} (\iota_r^* Y)_n$$

and if $\theta = f \circ u : [m] \to [n]$, then

$$\begin{tikzcd}
(\iota_r^* Y)_n \arrow[d, "\theta^*"] & \mathrm{Aut}_{\Lambda_r^{op}}([n]) \times Y_n \arrow[l, "\sim"'] \arrow[d] \ni & (v^{op}, y) \arrow[d, mapsto] \\
(\iota_r^* Y)_m & \mathrm{Aut}_{\Lambda_r^{op}}([m]) \times Y_m \arrow[l, "\sim"'] \ni & ((f^\# v \circ u)^{op}, (v_\# f)^* y)
\end{tikzcd}$$

Lemma The canonical map

$$\iota_r^* \, sd_r Y \longrightarrow sd_r \, \iota^* Y$$

is an isomorphism.

pf The can. map is induced from the functor $\iota_r / [n] \longrightarrow \iota / sd_r([n])$ which takes the object $([m], \theta : [m] \to [n])$ to the object $(sd^r([m]), sd^r\theta : sd^r([m]) \to sd^r([n]))$. This is the inclusion of a full subcategory and maps the cofinal subcat. $\mathrm{Aut}_{\Delta_r^{op}}([n])$ of $\iota_r/[n]$ isomorphically onto the cofinal subcat. $\mathrm{Aut}_{\Delta^{op}}(sd^r([n]))$ of $\iota/sd_r([n])$. //

Cor There is a can. isom. of simpl. sets

$$sd_r(\Delta[1]/\partial\Delta[1]) \xrightarrow{\sim} \iota_{r*}\iota_r^*(pt).$$

pf $\iota_{r*}\iota_r^*(pt) = \iota_{r*}\iota_r^* sd_r(pt)$

$$\xrightarrow{\sim} \iota_{r*} sd_r \iota^*(pt) = sd_r \iota_* \iota^*(pt)$$

and $\Delta[1]/\partial\Delta[1] \xrightarrow{\sim} \iota_* \iota^*(pt)$, $id_{[1]} \mapsto t_1$. //

start → Prop There is a can. (non-simpl.) homeo.

$$|\iota_{r*}\iota_r^* Y| \underset{\sim}{\xrightarrow{D_r}} |\iota_{r*}\iota_r^*(pt) \times Y|$$

pf Let $pr : \Delta_r \to \Delta$ be the functor given by the identity on Δ and by

$$\begin{array}{ccc} \mathrm{Aut}_{\Delta_r}([n]) & \xrightarrow{pr} & \mathrm{Aut}_{\Delta}([n]) \\ \psi & & \psi \\ \tau_{r,n} & \longmapsto & \tau_n \end{array}$$

Then ℓ_r is defined by

$$\ell_r(v^{op}, y; z) = ((v^{-1})^{op}, y; pr(v)^* z).$$

Check this is well-defined. //

Let X be a Δ_r^{op}-set and define

$$|\iota_{r*}\iota_r^*\iota_{r*}(pt)| \times |\iota_{r*}X| \xrightarrow{\mu_{r,X}} |\iota_{r*}X|$$

to be the composite

$$|\iota_{r*}\iota_r^*\iota_{r*}(pt)| \times |\iota_{r*}X|$$

$$\xrightarrow[\sim]{h_r \times id} |\iota_{r*}\iota_r^*\iota_{r*}(pt)| \times |\iota_{r*}X|$$

$$\xleftarrow[\sim]{h_r} |\iota_{r*}\iota_r^*\iota_{r*}X| \xrightarrow{\varepsilon} |\iota_{r*}X|.$$

<u>Lemma</u> The vertex $t_{r,0}^i \in |\iota_{r*}\iota_r^*\iota_{r*}(pt)|$ acts on $|\iota_{r*}X|$ as the simplicial map

$$(\iota_{r*}X)_n \xrightarrow{t_{r,n}^{(n+1)i}} (\iota_{r*}X)_n .$$

Pf Let $f : [n] \to [0]$ be the unique map, let $x \in X_n$ and $z \in \Delta^n$. Then

$$|i_{r*} i_r^* i_{r*}(pt)| \xrightarrow{e_r} |i_{r*} i_r^* i_{r*}(pt)|$$

$$(t_{r,0}^{i} ; f_* z) \longmapsto (t_{r,0}^{-i} ; f_* z)$$

$$|i_{r*} i_r^* i_{r*}(pt)| \times |i_{r*} X| \xleftarrow{pr_1 \times pr_2} |i_{r*} i_r^* i_{r*}(pt) \times i_{r*} X|$$

$$((f^* t_{r,0}^{-i} ; z), (x ; z)) \longleftarrow\!\shortmid (f^* t_{r,0}^{-i}, x ; z)$$

$$\qquad \| \qquad\qquad\qquad\qquad\qquad \|$$

$$((t_{r,0}^{-i} ; f_* z), (x ; z)) \qquad\qquad (t_{r,n}^{-(n+1)i}, x ; z) \longmapsto$$

$$\xleftarrow{e_r} |i_{r*} i_r^* i_{r*} X| \xrightarrow{\varepsilon} |i_{r*} X|$$

$$(t_{r,n}^{(n+1)i}, x ; pr(t_{r,n}^{(n+1)i}) z)$$

$$\|$$

$$(t_{r,n}^{(n+1)i}, x ; z) \longmapsto (t_{r,n}^{(n+1)i} x ; z). \qquad //$$

Prop Let X be a cyclic set. Then the following diagram commutes.

$$\begin{tikzcd}
{|sd_r \Delta[1]/\partial\Delta[1]| \times |\iota_{r*} sd_r X|} \arrow[r, "\mu_{r,X}"] \arrow[d, "\sim"', "D_r \times D_r"] & {|\iota_{r*} X|} \arrow[d, "\sim"', "D_r"] \\
{|\Delta[1]/\partial\Delta[1]| \times |\iota_* X|} \arrow[r, "\mu_X"] & {|\iota_* X|}
\end{tikzcd}$$

pf The essential thing to show is that the following diagr. commutes.

$$\begin{tikzcd}
{|\iota_{r*}\iota_r^*\iota_{r*}(pt) \times \iota_{r*} sd_r X|} \arrow[d, "\sim"] \arrow[r, phantom, "\circlearrowleft", shift right=8] & {|\iota_{r*}\iota_r^*\iota_{r*} sd_r X|} \arrow[l, "e_r"', "\sim"] \arrow[r, "\varepsilon"] \arrow[d, "\sim"] \arrow[r, phantom, "\circlearrowleft", shift right=8] & {|\iota_{r*} sd_r X|} \arrow[d, equal] \\
{|sd_r(\iota_*\iota^*\iota_*(pt) \times \iota_* X)|} \arrow[d, "D_r"] & {|sd_r(\iota_*\iota^*\iota_* X)|} \arrow[l, "e_r"', "\sim"] \arrow[r, "\varepsilon"] \arrow[d, "D_r"] \arrow[r, phantom, "\circlearrowleft", shift right=8] & {|sd_r \iota_* X|} \arrow[d, "D_r"] \\
{|\iota_*\iota^*\iota_*(pt) \times \iota_* X|} & {|\iota_*\iota^*\iota_* X|} \arrow[l, "e"', "\sim"] \arrow[r, "\varepsilon"] & {|\iota_* X|}
\end{tikzcd}$$

Only for the lower left-hand sq. is there something to check.

$$D_r\, e_r(v^{op}, x; z) = ((v^{-1})^{op}, x; d_r(pr(v)^* z))$$

$$e\, D_r(v^{op}, x; z) = ((v^{-1})^{op}, x; v^* d_r(z)).$$

Since elem. in the kernel of pr fixes the image of $d_r : \Delta^{n-1} \to \Delta^{rn-1}$,

$$v^* d_r(z) = d_r(pr(v)^* z). \qquad //$$

$$\begin{tikzcd} & \Delta \arrow[dl, "i_r"'] \arrow[dr, "i"] & \\ \Delta_r \arrow[rr, "p_r"] & & \Delta \end{tikzcd}$$

$$\left.\begin{aligned} i_* &= i_{r*}\, p_{r*} \\ i^* &= p_r^*\, i_r^* \end{aligned}\right\} \leadsto$$

$$i_{r*} i_r^* \xrightarrow{\eta} i_{r*} p_{r*} p_r^* i_r^* = i_* i^*$$

<u>Lemma</u> Let X be a cyclic set. Then the following diagr. commutes.

$$\begin{array}{ccc} |i_{r*} i_r^* i_{r*}(pt)| \times |i_{r*} p_{r*} X| & \xrightarrow{\mu_r} & |i_{r*} p_{r*} X| \\ \Big\downarrow \eta \times id & & \Big\| \\ |i_* i^* i_*(pt)| \times |i_* X| & \xrightarrow{\mu} & |i_* X| \end{array}$$

//

<u>Lemma</u> The unit of the adjunction (p_r^*, p_{r*}) induces an iso. of simpl. sets

$$i_{r*} i_r^*(pt)/C_r \xrightarrow[\sim]{\eta} i_* i^*(pt)$$

//

k commutative (ground) ring

A (not necessarily comm.) unital k-algebra

M (symmetric) A-A-bimodule.

Simpl. k-module

$$HH(A;M) = HH(A/k;M)$$

with n-simpl.

$$HH(A/k;M)_n = M \otimes \underbrace{A \otimes \cdots \otimes A}_{n}$$

and simpl. structure maps

$$\begin{aligned} d_i(m \otimes a_1 \otimes \cdots \otimes a_n) &= ma_1 \otimes a_2 \otimes \cdots \otimes a_n & i=0 \\ &= m \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & 0<i<n \\ &= a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1} & i=n \end{aligned}$$

$$s_i(m \otimes a_1 \otimes \cdots \otimes a_n) = m \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n \qquad 0 \le i \le n$$

If $M = A$, then $HH(A/k) = HH(A/k;A)$ is a cyclic k-module with

$$t_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_n \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}.$$

<u>ex</u> $A^{\otimes r}$ k-alg. w. comp.-wise mult.

${}_t(A^{\otimes r})$ $A^{\otimes r}$ – $A^{\otimes r}$ – bimodule with

left $A^{\otimes r}$ – mod. str.

$$(a_1 \otimes \cdots \otimes a_r) \cdot (a_1' \otimes \cdots \otimes a_r')$$

$$= (a_r a_1' \otimes a_1 a_2' \otimes \cdots \otimes a_{r-1} a_r')$$

and right $A^{\otimes r}$ – mod. str.

$$(a_1' \otimes \cdots \otimes a_r') \cdot (a_1 \otimes \cdots \otimes a_r)$$

$$= (a_1' a_1 \otimes \cdots \otimes a_r' a_r).$$

Isomorphism of simpl. k-module

$$\mathrm{sd}_r \, HH(A) \xrightarrow{\sim} HH(A^{\otimes r}, {}_t(A^{\otimes r}))$$

given by

$$a_{0,1} \otimes a_{1,1} \otimes \cdots \otimes a_{n,1} \otimes \cdots \otimes a_{0,r} \otimes a_{1,r} \otimes \cdots \otimes a_{n,r}$$

$$\mapsto a_{0,1} \otimes a_{0,2} \otimes \cdots \otimes a_{0,r} \otimes \cdots \otimes a_{n,1} \otimes a_{n,2} \otimes \cdots \otimes a_{n,r}.$$

Cyclic permutation of the tensor-factors defines a simpl. C_r - action on $HH(A^{\otimes r}, {}_t(A^{\otimes r}))$. This corresponds to the simplicial

G_r-action on $sd_r HH(A)$. It follows that the induced action on $HH_*(A^{\otimes r}, {}_t(A^{\otimes r}))$ is trivial. Indeed, we have canonical isom.

$$HH_*(A^{\otimes r}, {}_t(A^{\otimes r})) \approx \pi_* |HH(A^{\otimes r}, {}_t(A^{\otimes r}))|$$

$$\xleftarrow{\sim} \pi_* | sd_r HH(A) |$$

and the G_r-action on $|sd_r HH(A)|$ extends to an action by the circle. Hence the induced action on htpy. grps. is trivial. //

$A^{op} = A$ with mult. $a * a' = a'a$.

$A^e = A \otimes A^{op}$

An A-A-bimod. is the "same" as a left (or right) A^e-mod.:

$(a \otimes a').m = ama'$

$(m.(a \otimes a') = a'ma)$.

Canonical iso.

$$HH_0(A; M) \xrightarrow{\sim} M \otimes_{A^e} A$$

$$\text{class of } m \longmapsto m \otimes 1$$

Two-sided bar-construction :

$$B(A,A,A)_n = A \otimes \cdots \otimes A \quad (n+2 \text{ factors})$$

simpl. A-A-bimod. with

$$d_i(a_0 \otimes \cdots \otimes a_{n+1}) = a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \quad 0 \le i \le n$$

$$s_i(a_0 \otimes \cdots \otimes a_{n+1}) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_{n+1} \quad 0 \le i \le n.$$

The "remaining degeneracy"

$$s_{-1}(a_0 \otimes \cdots \otimes a_{n+1}) = 1 \otimes a_0 \otimes \cdots \otimes a_{n+1}$$

gives a contracting chain homotopy of the assoc. chain cx. So

$$B(A,A,A)_* \xrightarrow{\varepsilon} A$$

$$a_0 \otimes a_1 \longmapsto a_0 a_1$$

is a resolution of the (left) A^e-mod. A. If A is flat (resp. proj., resp. free) over k, then $B(A,A,A)_n$ is flat (resp. proj., resp. free) over A^e. Isom. of simpl. k-modules

$$M \underset{A^e}{\otimes} B(A,A,A) \xrightarrow{\sim} HH(A;M)$$

$$m \otimes (a_0 \otimes \cdots \otimes a_{n+1}) \longmapsto a_{n+1} m a_0 \otimes a_1 \otimes \cdots \otimes a_n .$$

Prop If A is flat over k, then

$$HH_*(A;M) = \mathrm{Tor}_*^{A^e}(M,A).$$ //

$$\mathrm{Tor}_*^{A^e}(A,A) \otimes \mathrm{Tor}_*^{A^e}(A,A) \to \mathrm{Tor}_*^{A^e}(A,A)$$

$$H_*(A \otimes_{A^e} P) \otimes H_*(A \otimes_{A^e} P) \to H_*(A \otimes A \otimes_{A^e \otimes A^e} P \otimes P)$$

$$\xrightarrow{\mu \otimes \tilde{\mu}} H_*(A \otimes_{A^e \otimes A^e} P) \to H_*(A \otimes_{A^e} P).$$

$$\begin{tikzcd} P \otimes P \arrow[r, "\tilde{\mu}"] \arrow[d, "\varepsilon \otimes \varepsilon"] & P \arrow[d, "\varepsilon"] \\ A \otimes A \arrow[r, "\mu"] & A \end{tikzcd}$$

$$B(A,A,A) \otimes B(A,A,A) \xrightarrow{\tilde{\mu}} B(A,A,A)$$

$$a_0 \otimes \cdots \otimes a_{n+1} \otimes a_0' \otimes \cdots \otimes a_{n+1}' \mapsto a_0 a_0' \otimes \cdots \otimes a_{n+1} a_{n+1}'.$$

<u>Example</u> $A = k[x_1, \ldots, x_n]$

$P = A^e \otimes \Lambda\{y_1, \ldots, y_n\}$

$dy_i = x_i \otimes 1 - 1 \otimes x_i$

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\varepsilon"] & B(A,A,A) \arrow[d, "\varepsilon"] \\ A \arrow[r, equal] & A \end{tikzcd}$$

$$y_i \mapsto s_{-1}(dy_i) = 1 \otimes x_i \otimes 1 - 1 \otimes 1 \otimes x_i$$

$$A \otimes_{A^e} P \longrightarrow A \otimes_{A^e} B(A,A,A) \xrightarrow{\text{can}} HH(A)$$

$$1 \otimes_{A^e} y_i \longmapsto 1 \otimes_{A^e} (1 \otimes x_i \otimes 1) \longmapsto 1 \otimes x_i - x_i \otimes 1$$
$$- 1 \otimes_{A^e} (1 \otimes 1 \otimes x_i)$$

$$B(x_i) = 1 \otimes x_i \underset{\text{homologous}}{\sim} 1 \otimes x_i - x_i \otimes 1$$

so

$$HH_*(k[x_1, \dots, x_n])$$

$$\simeq k[x_1, \dots, x_n] \otimes \Lambda \{B(x_1), \dots, B(x_n)\}.$$ //

Canonical map

$$\Omega^*_{A/k} \xrightarrow{\lambda} HH_*(A/k)$$

$$a_0 da_1 \dots da_n \longmapsto \text{class of } a_0 Ba_1 \dots Ba_n.$$

Isomorphism of $A = k[x_1, \dots, x_n]$.

$(dB + Bd = 0.)$

Symmetric spectrum E:

- (left) Σ_n-space E_n, $n \geq 0$,

- maps

$$S^m \wedge E_n \xrightarrow{\sigma_{m,n}} E_{m+n}, \quad m,n \geq 0$$

that are $\Sigma_m \times \Sigma_n$-equiv. s.t. $\sigma_{0,n}$ is the can. iso. and s.t.

$$\begin{tikzcd}
S^k \wedge S^m \wedge E_n \arrow[r, "\mathrm{id} \wedge \sigma_{m,n}"] \arrow[d, "\sim"', "\mathrm{can}"] \arrow[dr, phantom, "\circlearrowleft"] & S^k \wedge E_{m+n} \arrow[d, "\sigma_{k,m+n}"] \\
S^{k+m} \wedge E_n \arrow[r, "\sigma_{k+m,n}"] & E_{k+m+n}
\end{tikzcd}$$

//

Topological (index) category $\mathcal{J}$:

$\mathrm{ob}\,\mathcal{J} = \mathbb{N}_0$ = non-neg. integers.

$$\underline{\mathrm{Hom}}_{\mathcal{J}}(n, m+n) = (\Sigma_{m+n})_+ \underset{\Sigma_m}{\wedge} S^m$$

composition:

$$\underline{\mathrm{Hom}}_{\mathcal{J}}(m+n, k+m+n) \wedge \underline{\mathrm{Hom}}_{\mathcal{J}}(n, m+n)$$

$$\xrightarrow{\circ} \underline{\mathrm{Hom}}_{\mathcal{J}}(n, k+m+n)$$

$$(\Sigma_{k+m+n})_+ \underset{\Sigma_k}{\wedge} S^k \wedge (\Sigma_{m+n})_+ \underset{\Sigma_m}{\wedge} S^m$$

$$\longrightarrow (\Sigma_{k+m+n})_+ \underset{\Sigma_{k+m}}{\wedge} S^{k+m}$$

Then a symmetric spectrum is a topological functor

$$\mathcal{J} \xrightarrow{E} \mathrm{Top}.$$

Level model structure (or objectwise fibration model structure):

weak equiv. = objectwise weak equiv.

fibrations = objectwise fibrations.

cofibrations = LLP w.r.t. trivial fibrations.

Cofibrantly generated: sets

$I: \quad S^{n-1}_+ \xhookrightarrow{i} D^n_+$

$J: \quad D^n_+ \xhookrightarrow{j} (D^n \times [0,1])_+$

$FI: \quad F_n i \; , \; i \in I \; ,$

$FJ: \quad F_n j \; , \; j \in J \; .$

$$\begin{tikzcd}
\bigvee_{k+l+m=n} (\Sigma_{k+l+m})_+ \wedge_{\Sigma_k \times \Sigma_l \times \Sigma_m} (E_k \wedge S^l \wedge E'_m) \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \\
\bigvee_{p+q=n} (\Sigma_{p+q})_+ \wedge_{\Sigma_p \times \Sigma_q} (E_p \wedge E'_q) \arrow[d] \\
(E \wedge E')_n
\end{tikzcd}$$

$$f(\alpha, e, x, e') = (\alpha \circ \rho_{k,l}^{-1}, \sigma_{l,k}(x,e), e')$$

$$g(\alpha, e, x, e') = (\alpha, e, \sigma'_{l,m}(x, e')).$$

$$\rho_{k,l}(i) = \begin{cases} i+l & , 1 \leq i \leq k \\ i-k & , k+1 \leq i \leq k+l \\ i & , k+l+1 \leq i \leq k+l+m \end{cases}$$

$$\mathrm{Top} \underset{ev_n}{\overset{F_n}{\rightleftarrows}} \mathrm{Top}^{\mathcal{J}}$$

$$(F_n X)_m = \operatorname*{colim}_{\iota/m} X$$

$$= (\Sigma_m)_+ \underset{\Sigma_{m-n}}{\wedge} S^{m-n} \wedge X$$

$$\underline{\mathrm{Hom}}_{\mathrm{Top}^{\mathcal{J}}}(F_n X, E) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathrm{Top}}(X, E_n) .$$

$$\begin{array}{ccc} (F_{n+1} S^1)_m & \xrightarrow{\lambda_n} & (F_n S^0)_m \\ \| & & \| \\ (\Sigma_m)_+ \underset{\Sigma_{m-n-1}}{\wedge} S^{m-n-1} \wedge S^1 & \xrightarrow{proj} & (\Sigma_m)_+ \underset{\Sigma_{m-n}}{\wedge} S^{m-n} \end{array}$$

Obtain stable model structure by localizing level model str. w.r.t. the following set of maps

$$A = \{ F_{n+1} S^1 \xrightarrow{\lambda_n} F_n S^0 \mid n \geq 0 \}.$$

Explain localization:

Def (1) An object W is A-<u>local</u> if (W is fibrant and) for every map λ_n in A, the induced map

$$\underline{Hom}_{Top^{\mathcal{J}}}(F_n S^0, W) \xrightarrow{\lambda_n^*} \underline{Hom}_{Top^{\mathcal{J}}}(F_{n+1} S^1, W)$$

is a weak equivalence.

(2) A map $f : E \to E'$ is an A-<u>local</u>

equivalence if for every A-local obj. W, the induced map

$$\underline{\mathrm{Hom}}_{\mathrm{Top}^{\Sigma}}(E', W) \xrightarrow{f^*} \underline{\mathrm{Hom}}_{\mathrm{Top}^{\Sigma}}(E, W)$$

is a weak equivalence. //

Thm The category Top^{Σ} of symmetric spectra is a cofibrantly generated topological model category with

weak equiv. = A-local weak equiv.
cofibrations = level cofibrations
fibrations = RLP w.r.t. trivial cof.

Generators:

$$FI: \quad F_m((S^{n-1})_+) \to F_m((D^n)_+) \qquad m, n \geq 0$$

$$K: \quad F_m((D^n)_+) \to F_m((D^n \times I)_+) \qquad m, n \geq 0$$

and the maps (i_m, k_n), $m, n \geq 0$, defined as follows:

$$
\begin{tikzcd}
F_{n+1}S^1 \arrow[r, "\lambda_n"] \arrow[d, shift right, "i_0"'] \arrow[dr, phantom, "\text{p.o.}"] & F_n S^0 \arrow[d, shift right] \\
{[0,1]_+ \wedge F_{n+1}S^1} \arrow[r] \arrow[u, shift right, "r"'] & \mathrm{Cyl}(\lambda_n) \arrow[u, shift right, "r_n"'] \\
F_{n+1}S^1 \arrow[u, "i_1"] \arrow[ur, "k_n"'] \arrow[uu, bend left=60, "\mathrm{id}"] &
\end{tikzcd}
$$

r_n = level equiv.

k_n = cofibration

$$
\begin{tikzcd}
(S^{m-1})_+ \wedge F_{n+1}S^1 \arrow[r, "\mathrm{id}\wedge k_n"] \arrow[d, "i_m \wedge \mathrm{id}"] \arrow[dr, phantom, "\text{p.o.}"] & (S^{m-1})_+ \wedge \mathrm{Cyl}(\lambda_n) \arrow[d] \arrow[dd, bend left=60, "i_m \wedge \mathrm{id}"] \\
(D^m)_+ \wedge F_{n+1}S^1 \arrow[r] \arrow[dr, bend right, "\mathrm{id}\wedge k_n"'] & (D^m)_+ \wedge F_{n+1}S^1 \cup (S^{m-1})_+ \wedge \mathrm{Cyl}(\lambda_n) \arrow[d, dashed, "(i_m; k_n)"] \\
& (D^m)_+ \wedge \mathrm{Cyl}(\lambda_n)
\end{tikzcd}
$$

The fibrant objects w.r.t. the stable (or A-local) model str. are the A-local objects:

$$\begin{array}{ccc}
\underline{\mathrm{Hom}}_{\mathcal{T}op^{\mathcal{J}}}(F_n S^0, E) & \xrightarrow{\lambda_n^*} & \underline{\mathrm{Hom}}_{\mathcal{T}op^{\mathcal{J}}}(F_{n+1} S^1, E) \\
\downarrow \sim & & \downarrow \sim \\
\underline{\mathrm{Hom}}_{\mathcal{T}op}(S^0, E_n) & & \underline{\mathrm{Hom}}_{\mathcal{T}op}(S^1, E_{n+1}) \\
\| & & \| \\
E_n & \longrightarrow & \Omega E_{n+1}
\end{array}$$

adjoint of the map $\sigma'_{n,1}$ given by

$$\begin{array}{ccc}
E_n \wedge S^1 & \xrightarrow{\sigma'_{n,1}} & E_{n+1} \\
\downarrow tw & & \sim \downarrow \rho_{n,1} \\
S^1 \wedge E_n & \xrightarrow{\sigma_{1,n}} & E_{1+n}
\end{array}$$

So the fibrant objects are exactly the Ω-spectra.

Define homotopy groups

$$\pi_q(X) := \operatorname*{colim}_n \pi_{q+n}(X_n)$$

with str. maps

$$\pi_{q+n} X_n \xrightarrow{\tilde{\sigma}'_{n+1}} \pi_{q+n} \Omega X_{n+1} \xrightarrow{\sim} \pi_{q+n+1} X_{n+1}.$$

If E is an Ω-spectrum, then the maps in the limit system are isom. so we have a can. isom.

$$\pi_q(E_0) \xrightarrow{\sim} \pi_q(E).$$

Prop Let $f: X \to Y$ be a map of symm. spectra and suppose that for all integers q, the induced map

$$\pi_q(X) \xrightarrow{f_*} \pi_q(Y)$$

is an isom. Then f is a stable equivalence.

pf There is a can. isom.

$$F_m A \wedge F_n B \xrightarrow{\sim} F_{m+n}(A \wedge B).$$

Define

$$QX = \underset{n}{\operatorname{hocolim}}\ F(F_n S^n, X)$$

with str. maps induced by

$$F_{n+1}S^{n+1} \xleftarrow{\sim} F_n S^n \wedge F_1 S^1 \xrightarrow{\mathrm{id}\wedge\lambda_1} F_n S^n \wedge F_0 S^0 \xleftarrow{\sim} F_n S^n .$$

Claim: $\pi_q((QX)_m) \xrightarrow{\sim} \pi_{q-m}(X)$.

$$\pi_q(QX)_m = \pi_q\ \underset{n}{\operatorname{hocolim}}\ F(F_n S^n, X)_m$$

$$\xleftarrow{\sim} \underset{n}{\operatorname{colim}}\ \pi_q F(F_n S^n, X)_m .$$

$\operatorname{Hom}_{Top}(A, F(F_n S^n, X)_m)$

$\approx \operatorname{Hom}_{Top^{\mathcal{J}}}(F_m A, F(F_n S^n, X))$

$\approx \operatorname{Hom}_{Top^{\mathcal{J}}}(F_m A \wedge F_n S^n, X)$

$\approx \operatorname{Hom}_{Top^{\mathcal{J}}}(F_{m+n}(A \wedge S^n), X)$

$\approx \operatorname{Hom}_{Top}(A \wedge S^n, X_{m+n})$

$\approx \operatorname{Hom}_{Top}(A, \Omega^n X_{m+n})$, So Qf is a level equivalence. Moreover, if E is an Ω-spectrum, then

$$E \xrightarrow{\iota} QE$$

is a level equivalence. So

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathrm{Top}^{\mathcal{J}}}(X, E) \arrow[rr, "\iota_*", "\sim"'] \arrow[dr, "Q"'] & & \underline{\mathrm{Hom}}_{\mathrm{Top}^{\mathcal{J}}}(X, QE) \\
& \underline{\mathrm{Hom}}_{\mathrm{Top}^{\mathcal{J}}}(QX, QE) \arrow[ur, "\iota^*"'] &
\end{tikzcd}$$

It follows that

$$\pi_* \underline{\mathrm{Hom}}_{\mathrm{Top}^{\mathcal{J}}}(Y, E) \xrightarrow{f^*} \pi_* \underline{\mathrm{Hom}}_{\mathrm{Top}^{\mathcal{J}}}(X, E)$$

is a retract of

$$\pi_* \underline{\mathrm{Hom}}_{\mathrm{Top}^{\mathcal{J}}}(QY, QE) \xrightarrow{Qf^*} \pi_* \underline{\mathrm{Hom}}_{\mathrm{Top}^{\mathcal{J}}}(QX, QE).$$

Since Qf is a level equivalence, Qf^* is an isom. Hence so is f^*. This is true for every Ω-spectrum E. So f is a stable equivalence. //

Is every stable equivalence a π_*-isom. ? This is true if and only if the maps

$$F_{n+1} S^1 \xrightarrow{\lambda_n} F_n S^0$$

are π_*-isom. (The "if" part is not completely trivial.) Exercise:

$$\pi_*(f) \text{ isom.} \iff \pi_*(\Sigma f) \text{ isom.}$$

Consider

$$\begin{tikzcd}
((F_{n+1}S^1)\wedge S^n)_m \arrow[rr, "\lambda_n \wedge \mathrm{id}"] \arrow[d, equal] & & ((F_n S^0)\wedge S^n)_m \\
(\Sigma_m)_+ \wedge_{\Sigma_{m-n-1}} S^{m-n-1} \wedge S^1 \wedge S^n \arrow[rr, "\mathrm{proj}"] & & (\Sigma_m)_+ \wedge_{\Sigma_{m-n}} S^{m-n} \wedge S^n \\
(\Sigma_m / \Sigma_{m-n-1})_+ \wedge S^m \arrow[u, "\sim"'] \arrow[rr, "\mathrm{proj} \wedge \mathrm{id}"] & & (\Sigma_m / \Sigma_{m-n})_+ \wedge S^m \arrow[u, "\sim"']
\end{tikzcd}$$

It follows that

$$\pi_0((F_{n+1}S^1)\wedge S^n) \longrightarrow \pi_0((F_n S^0)\wedge S^n)$$

is not an isom.

However, the corresponding projection map for orthogonal groups

$$\left(O(m)/O(m-n-1)\right)_+ \wedge S^n \longrightarrow \left(O(m)/O(m-n)\right)_+ \wedge S^n$$

is $(2m-n-1)$-connected. So if we replace Σ_n by $O(n)$ everywhere we get a (stable model) category of orthogonal spectra, where the weak equivalences are the π_*-isom.

Define top'l index category $\mathcal{J}_O$:

$$\mathrm{ob}\, \mathcal{J}_O = \mathbb{N}_0 = \text{non-neg. integers.}$$

$$\underline{\mathrm{Hom}}_{\mathcal{J}_O}(n, m+n) = O(m+n)_+ \underset{O(m)}{\wedge} S^m$$

composition

$$\underline{\mathrm{Hom}}_{\mathcal{J}_O}(m+n, k+m+n) \wedge \underline{\mathrm{Hom}}_{\mathcal{J}_O}(n, m+n)$$

$$\xrightarrow{\circ} \underline{\mathrm{Hom}}_{\mathcal{J}_O}(n, k+m+n)$$

given by

$$O(k+m+n)_+ \underset{O(k)}{\wedge} S^k \wedge O(m+n)_+ \underset{O(m)}{\wedge} S^m$$

$$\longrightarrow O(k+m+n)_+ \underset{O(k+m)}{\wedge} S^{k+m} .$$

Then an orthogonal spectrum is a continuous functor

$$\mathcal{J}_O \xrightarrow{E} Top .$$

View $\Sigma_n \subset O(n)$ as the subgroup that permutes the standard basis of $\mathbb{R}^n$. Then the category $\mathcal{J}_\Sigma$, which indexes symmetric spectra, is canonically a (topological) subcategory of $\mathcal{J}_O$. So we have an adjoint pair of functors

$$Top^{\mathcal{J}_\Sigma} \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} Top^{\mathcal{J}_O} .$$

This is a Quillen equivalence:

(i) ι^* preserves (trivial) cofibrations.

(ii) ι_* preserves (trivial) fibrations.

(iii) a map $E \to \iota_* E'$, with E cofibrant and E' fibrant, is a w.e. if and only if the adjoint map $\iota^* E \to E'$ is a w.e.

This implies that the total derived functors

$$\mathrm{Ho}\,\mathrm{Top}^{\mathcal{J}_E} \underset{\mathbb{R}\iota_*}{\overset{\mathbb{L}\iota^*}{\rightleftarrows}} \mathrm{Ho}\,\mathrm{Top}^{\mathcal{J}_0}$$

is an adjoint equivalence of cat.

A symmetric ring spectrum is a monoid in the symmetric monoidal category of symmetric spectra and smash product. This amounts to the same a sequence of left Σ_n-spaces E_n together with unit and multiplication maps

$$S^n \xrightarrow{\eta_n} E_n$$

$$E_m \wedge E_n \xrightarrow{\mu_{m,n}} E_{m+n}$$

that are Σ_n and $\Sigma_m \times \Sigma_n$-equiv., resp., and s.t. the following unity, associativity, and centrality of unit diagrams commute

$$\begin{tikzcd} S^m \wedge S^n \arrow[r, "\eta_m \wedge \eta_n"] \arrow[d, "\sim"', "can"] & E_m \wedge E_n \arrow[d, "\mu_{m,n}"] \\ S^{m+n} \arrow[r, "\eta_{m+n}"] & E_{m+n} \end{tikzcd}$$

$$\begin{tikzcd} E_k \wedge E_m \wedge E_n \arrow[r, "\mu_{k,m} \wedge id"] \arrow[d, "id \wedge \mu_{m,n}"] & E_{k+m} \wedge E_n \arrow[d, "\mu_{k+m,n}"] \\ E_k \wedge E_{m+n} \arrow[r, "\mu_{k,m+n}"] & E_{k+m+n} \end{tikzcd}$$

$$\begin{tikzcd}
S^m \wedge E_n \arrow[r, "\eta_m \wedge id"] \arrow[d, "tw"] & E_m \wedge E_n \arrow[r, "\mu_{m,n}"] & E_{m+n} \arrow[d, "\rho_{m,n}"] \\
E_n \wedge S^m \arrow[r, "id \wedge \eta_m"] & E_n \wedge E_m \arrow[r, "\mu_{n,m}"] & E_{n+m}
\end{tikzcd}$$

and s.t.

$$\begin{tikzcd}
S^0 \wedge E_n \arrow[r, "\eta_0 \wedge id"] \arrow[dr, "can", "\sim"'] & E_0 \wedge E_n \arrow[d, "\mu_{0,n}"] \\
& E_n
\end{tikzcd}$$

A symm. ring spectrum is commutative if and only if the following diagram commutes.

$$\begin{tikzcd}
E_m \wedge E_n \arrow[r, "\mu_{m,n}"] \arrow[d, "tw"] & E_{m+n} \arrow[d, "\rho_{m,n}"] \\
E_n \wedge E_m \arrow[r, "\mu_{n,m}"] & E_{n+m}
\end{tikzcd}$$

<u>ex</u> 1) A monoid Π in the symmetric monoidal cat. of pointed spaces and smash product determines a symmetric ring spectrum with n'th space $S^n \wedge \Pi$

and with structure maps

$$S^n \xrightarrow[\sim]{can} S^n \wedge S^0 \xrightarrow{id \wedge \eta} S^n \wedge \overline{\Pi}$$

$$(S^m \wedge \overline{\Pi}) \wedge (S^n \wedge \overline{\Pi}) \xrightarrow[\sim]{can} S^{m+n} \wedge \overline{\Pi} \wedge \overline{\Pi}$$

$$\xrightarrow{id \wedge \mu} S^{m+n} \wedge \overline{\Pi}.$$

It is commutative iff $\overline{\Pi}$ is.

ii) A ring A determines a symmetric ring spectrum $\tilde{A}$. Let

$$S^n_\cdot = \underbrace{\Delta[1]/\partial\Delta[1] \wedge \cdots \wedge \Delta[1]/\partial\Delta[1]}_{n}$$

Then

$$\tilde{A}_n = |[k] \mapsto A\{S^n_k\}/A\{s_{0,k}\}|$$

with str. maps

$$S^n \xrightarrow[\sim]{can} |[k] \mapsto S^n_k|$$

$$\xrightarrow{\eta} |[k] \mapsto A\{S^n_k\}/A\{s_{0,k}\}| = \tilde{A}_n$$

$$\tilde{A}_m \wedge \tilde{A}_n = |[k] \mapsto A\{S^m_k\}/A\{s_{0,k}\}| \wedge |[k] \mapsto A\{S^n_k\}/A\{s_{0,k}\}|$$

$$\xleftarrow[\sim]{pr_1 \wedge pr_2} \left| [k] \mapsto A\{S^m_k\}/A\{s_{0,k}\} \wedge A\{S^n_k\}/A\{s_{0,k}\} \right|$$

$$\xrightarrow{\mu} \left| [k] \mapsto A\{S^m_k \wedge S^n_k\}/A\{s_{0,k}\} \right|$$

$$\xrightarrow[\sim]{can} \left| [k] \mapsto A\{S^{m+n}_k\}/A\{s_{0,k}\} \right| = \tilde{A}_{m+n} .$$

It is comm. if A is.

Recall

$$\mathrm{Top}^I \underset{\Delta}{\overset{\operatorname{colim}_I}{\rightleftarrows}} \mathrm{Top}$$

$$(i \mapsto X) \longleftarrow\!\shortmid \; X$$

$$\mathrm{Top}^I \overset{\operatorname{hocolim}_I}{\rightleftarrows} \mathrm{Top}$$

$$(i \mapsto F(|i/I|_+, X)) \longleftarrow\!\shortmid \; X$$

In particular, have can. homeomorphism

$$\operatorname*{hocolim}_I (X_i \wedge Y) \xrightarrow{\sim} (\operatorname*{hocolim}_I X_i) \wedge Y .$$

Functoriality:

$$I \xrightarrow{g} I' \xrightarrow{X} Top \quad \Rightarrow$$

$$\operatorname*{hocolim}_{I} (X\circ g) \xrightarrow{can} \operatorname*{hocolim}_{I'} X .$$

Indeed this is the adjoint of the following map of I-diagrams.

$$X(g(i)) \xrightarrow{\eta} F(|g(i)/I'|_+, \operatorname*{hocolim}_{I'} X)$$

$$\xrightarrow{g^*} F(|i/I|_+, \operatorname*{hocolim}_{I'} X) .$$

//

Let E be a symmetric ring spectrum. We define a cyclic pointed space with k-simplices

$$THH(E)_k = \operatorname*{hocolim}_{I^{k+1}} G_k(E) .$$

The (index) category I is given by

ob I: $\underline{i} = \{1, 2, .., i\}$, $i \geq 1$, $\underline{0} = \emptyset$.

mor I: <u>all</u> injective maps.

Note that every morphism in I can be written (non-uniquely) as the composite of the standard inclusion

$$\begin{array}{ccc} \underline{i} & \xrightarrow{\iota} & \underline{i}' \\ S & \longmapsto & S \end{array}$$

and an automorphism. There is a functor

$$I^{k+1} \xrightarrow{G_k(E)} Top$$

that on objects is given by the function space

$$G_k(\underline{i_0}, \ldots, \underline{i_k}) = F(S^{i_0} \wedge \ldots \wedge S^{i_k}, E_{i_0} \wedge \ldots \wedge E_{i_k}).$$

To define G_k on morphisms, write $i_r' = i_r + j_r$, and let $(\underline{i_0}, \ldots, \iota, \ldots, \underline{i_k})$ be the morphism in I^{k+1} given by the standard incl. $\underline{i_r} \xrightarrow{\iota} \underline{i_r'}$ in the r'th factor. Then the induced map

$$G_k(\underline{i_0}, \ldots, \underline{i_r}, \ldots, \underline{i_k}) \longrightarrow G_k(\underline{i_0}, \ldots, \underline{i_r'}, \ldots, \underline{i_k})$$

takes the map

$$S^{i_0} \wedge \dots \wedge S^{i_r} \wedge \dots \wedge S^{i_k} \xrightarrow{f} E_{i_0} \wedge \dots \wedge E_{i_r} \wedge \dots \wedge E_{i_k}$$

to the composite

$$S^{i_0} \wedge \dots \wedge S^{i_r'} \wedge \dots \wedge S^{i_k} \xrightarrow[\sim]{can} S^{i_0} \wedge \dots \wedge S^{i_r} \wedge \dots \wedge S^{i_k} \wedge S^{j_r}$$

$$\xrightarrow{f \wedge \eta_{j_r}} E_{i_0} \wedge \dots \wedge E_{i_r} \wedge \dots \wedge E_{i_k} \wedge E_{j_r}$$

$$\xrightarrow{\sim} E_{i_0} \wedge \dots \wedge E_{i_r} \wedge E_{j_r} \wedge \dots \wedge E_{i_k}$$

$$\xrightarrow{\mu} E_{i_0} \wedge \dots \wedge E_{i_r'} \wedge \dots \wedge E_{i_k}.$$

The symmetric group $\Sigma_{i_r'}$ acts on $S^{i_r'}$ and $E_{i_r'}$ and on $G_k(\underline{i_0}, \dots, \underline{i_r'}, \dots, \underline{i_k})$ by conjugation. This defines G_k on morphisms. It is well-defined (independent of the way a general morphism $i_r \to i_r'$ is written as a composite of the standard incl. and an automorphism of i_r') since μ_{i_r, j_r} is $\Sigma_{i_r} \times \Sigma_{j_r}$-equivariant.

We next define the cyclic str. maps

$$THH(E)_k \xrightarrow{d_i} THH(E)_{k-1} \qquad 0 \leq i \leq k$$

$$THH(E)_k \xrightarrow{s_i} THH(E)_{k+1} \qquad 0 \leq i \leq k$$

$$THH(E)_k \xrightarrow{t_k} THH(E)_k .$$

The category $\underline{I}$ has a strict monoidal (but not symmetric monoidal) structure given by the concatenation functor

$$\begin{array}{ccc} \underline{I} \times \underline{I} & \xrightarrow{\sqcup} & \underline{I} \\ (\underline{i}, \underline{i}') & \longmapsto & \underline{i+i'} \\ \Big\downarrow (f, f') & & \Big\downarrow f \sqcup f' \\ (\underline{j}, \underline{j}') & \longmapsto & \underline{j+j'} \end{array}$$

$$(f \sqcup f')(s) = \begin{cases} f(s) & 1 \leq s \leq i \\ f'(s-i)+j & i+1 \leq s \leq i+i' . \end{cases}$$

Define functors of index categories

$$\underline{I}^{k+1} \xrightarrow{\partial_r} \underline{I}^k \qquad 0 \leq r \leq k$$

$$\underline{I}^{k+1} \xrightarrow{\delta_r} \underline{I}^{k+2} \qquad 0 \leq r \leq k$$

$$\underline{I}^{k+1} \xrightarrow{t_k} \underline{I}^{k+1}$$

by

$$\partial_r(\underline{i_0},\ldots,\underline{i_k}) = (\underline{i_0},\ldots,\underline{i_r}\amalg\underline{i_{r+1}},\ldots,\underline{i_k}) \qquad 0\le r<k$$

$$= (\underline{i_k}\amalg\underline{i_0},\underline{i_1},\ldots,\underline{i_{k-1}}) \qquad r=k$$

$$s_r(\underline{i_0},\ldots,\underline{i_k}) = (\underline{i_0},\ldots,\underline{i_r},\underline{0},\underline{i_{r+1}},\ldots,\underline{i_k}) \qquad 0\le r\le k$$

$$t_k(\underline{i_0},\ldots,\underline{i_k}) = (\underline{i_k},\underline{i_0},\underline{i_1},\ldots,\underline{i_{k-1}})$$

and natural transformations

$$G_k(E) \xrightarrow{\delta_r} G_{k-1}(E)\circ\partial_r \qquad 0\le r\le k$$

$$G_k(E) \xrightarrow{\sigma_r} G_{k+1}(E)\circ s_r \qquad 0\le r\le k$$

$$G_k(E) \xrightarrow{\tau_k} G_k(E)\circ t_k$$

Then the cyclic str. maps are the composites

$$d_r : \operatorname*{hocolim}_{I^{k+1}} G_k \xrightarrow{\delta_r} \operatorname*{hocolim}_{I^k} G_{k-1}\circ\partial_r \xrightarrow{\text{can}} \operatorname*{hocolim}_{I^k} G_{k-1}$$

$$s_r : \operatorname*{hocolim}_{I^{k+1}} G_k \xrightarrow{\sigma_r} \operatorname*{hocolim}_{I^{k+2}} G_{k+1}\circ s_r \xrightarrow{\text{can}} \operatorname*{hocolim}_{I^{k+2}} G_{k+1}$$

$$t_k : \operatorname*{hocolim}_{I^{k+1}} G_k \xrightarrow{\tau_k} \operatorname*{hocolim}_{I^{k+1}} G_k\circ t_k \xrightarrow{\text{can}} \operatorname*{hocolim}_{I^{k+1}} G_k .$$

The natural transf. ∂_r takes the map

$$S^{i_0}\wedge\cdots\wedge S^{i_k} \xrightarrow{f} E_{i_0}\wedge\cdots\wedge E_{i_k}$$

to the composite

$$S^{i_0}\wedge\cdots\wedge S^{i_r+i_{r+1}}\wedge\cdots\wedge S^{i_k} \xrightarrow[\sim]{can} S^{i_0}\wedge\cdots\wedge S^{i_r}\wedge S^{i_{r+1}}\wedge\cdots\wedge S^{i_k}$$

$$\xrightarrow{f} E_{i_0}\wedge\cdots\wedge E_{i_r}\wedge E_{i_{r+1}}\wedge\cdots\wedge E_{i_k}$$

$$\xrightarrow{\mu} E_{i_0}\wedge\cdots\wedge E_{i_r+i_{r+1}}\wedge\cdots\wedge E_{i_k}$$

if $0\le r<k$, and to the composite

$$S^{i_k+i_0}\wedge S^{i_1}\wedge\cdots\wedge S^{i_{k-1}} \xrightarrow[\sim]{can} S^{i_0}\wedge\cdots\wedge S^{i_{k-1}}\wedge S^{i_k}$$

$$\xrightarrow{f} E_{i_0}\wedge\cdots\wedge E_{i_{k-1}}\wedge E_{i_k}$$

$$\xrightarrow[\sim]{can} E_{i_k}\wedge E_{i_0}\wedge\cdots\wedge E_{i_{k-1}}$$

$$\xrightarrow{\mu} E_{i_k+i_0}\wedge E_{i_1}\wedge\cdots\wedge E_{k-1}$$

if $r=k$.

Let us check that $G_1(E) \xrightarrow{\delta_0} G_0(E) \circ \partial_0$ is a natural transformation. We can write every morphism $(\underline{i_0}, \underline{i_1}) \to (\underline{i_0}', \underline{i_1}')$ as a comp. $(\underline{i_0}, \underline{i_1}) \to (\underline{i_0}', \underline{i_1}) \to (\underline{i_0}', \underline{i_1}')$, so we can assume $\underline{i_0}' = \underline{i_0}$ or $\underline{i_1}' = \underline{i_1}$. Suppose $\underline{i_1}' = \underline{i_1}$ and write $\underline{i_0} \xrightarrow{\alpha} \underline{i_0}'$ as a comp. $\alpha = \sigma \circ \iota$ of the standard incl. and an automorphism. We show that

$$\begin{array}{ccc} G_1(E)(\underline{i_0}, \underline{i_1}) & \xrightarrow{\delta_0} & G_0(E)(\underline{i_0} \sqcup \underline{i_1}) \\ \downarrow (\alpha, id) & & \downarrow \alpha \sqcup id \\ G_1(E)(\underline{i_0}', \underline{i_1}) & \xrightarrow{\delta_0} & G_0(E)(\underline{i_0}' \sqcup \underline{i_1}) \end{array}$$

commutes. Let

$$S^{i_0} \wedge S^{i_1} \xrightarrow{f} E_{i_0} \wedge E_{i_1}$$

be an element of $G_1(E)(\underline{i_0}, \underline{i_1})$. Then the image of f by the two composites in the square above are given by the two composites in the outer square in the following diagram. The part ① commutes for trivial reasons. We consider the part ② separately.

$$
\begin{tikzcd}
S^{i_0'+i_1'} \arrow[dr, "(\sigma\sqcup id)^{-1}"] & & & & \\
& S^{i_0+i_1} \arrow[r, "can", "\sim"'] \arrow[d, "\rho^{-1}"] & S^{i_0}\wedge S^{i_1} \arrow[r, "can", "\sim"'] & S^{i_0}\wedge S^{i_1}\wedge S^{j_0} \arrow[d, "f\wedge id"] & \\
& S^{i_0+i_1+j_0} \arrow[d, "can", "\sim"'] & 1 & E_{i_0}\wedge E_{i_1}\wedge S^{j_0} \arrow[d, "id\wedge\eta"] & \\
& S^{i_0}\wedge S^{i_1}\wedge S^{j_0} \arrow[d, "f\wedge id"] & & E_{i_0}\wedge E_{i_1}\wedge E_{j_0} \arrow[d, "can", "\sim"'] & \\
& E_{i_0}\wedge E_{i_1}\wedge S^{j_0} \arrow[d, "id\wedge\eta"] & & E_{i_0}\wedge E_{j_0}\wedge E_{i_1} \arrow[d, "\mu\wedge id"] & \\
& E_{i_0}\wedge E_{i_1}\wedge E_{j_0} \arrow[d, "\mu\wedge id"] & 2 & E_{i_0+j_0}\wedge E_{i_1} \arrow[d, "\mu"] & \\
& E_{i_0+i_1}\wedge E_{j_0} \arrow[r, "\mu"] & E_{i_0+i_1+j_0} \arrow[r, "\rho"] & E_{i_0+j_0+i_1} \arrow[dr, "\sigma\sqcup id"] & \\
& & & & E_{i_0'+j_0+i_1'}
\end{tikzcd}
$$

$$
\begin{tikzcd}
E_{i_0}\wedge E_{i_1}\wedge S^{j_0} \arrow[rr, "\mathrm{id}\wedge\eta"] \arrow[ddrr, "\mathrm{can}", "\cong"'] \arrow[dddd, "\mathrm{id}\wedge\eta"'] & & E_{i_0}\wedge E_{i_1}\wedge E_{j_0} \arrow[rr, "\mu\wedge\mathrm{id}"] \arrow[dr, "\mathrm{id}\wedge\mu"] & & E_{i_0+i_1}\wedge E_{j_0} \arrow[rr, "\mu"] & & E_{i_0+i_1+j_0} \arrow[dddd, "\rho"] \\
 & & d & E_{i_0}\wedge E_{i_1+j_0} \arrow[urrr, "\mu"'] \arrow[dd, "\mathrm{id}\wedge\rho"] & a & & \\
 & & E_{i_0}\wedge S^{j_0}\wedge E_{i_1} \arrow[dd, "\eta"] & & & b & \\
 & c & & E_{i_0}\wedge E_{j_0+i_1} \arrow[drrr, "\mu"] & a & & \\
E_{i_0}\wedge E_{i_1}\wedge E_{j_0} \arrow[rr, "\mathrm{can}", "\cong"'] & & E_{i_0}\wedge E_{j_0}\wedge E_{i_1} \arrow[rr, "\mu\wedge\mathrm{id}"] \arrow[ur, "\mathrm{id}\wedge\mu"] & & E_{i_0+j_0}\wedge E_{i_1} \arrow[rr, "\mu"] & & E_{i_0+j_0+i_1}
\end{tikzcd}
$$

a : assoc. of μ

b : μ is $\Sigma_{i_0}\times\Sigma_{i_1+j_0}$ - equiv.

c : naturality of "can".

d : centrality of unit η.

This shows that δ_0 is natural in the first variable. The proof that δ_0 is natural in second variable is easier.

Next, we verify the simplicial identity

$$d_i s_j = s_{j-1} d_i \qquad 0 \le i < j \le k.$$

We must show that the outer square in the following diagram commutes.

$$\begin{tikzcd}
\underset{I^{k+1}}{\mathrm{hocolim}}\, G_k \arrow[r, "\sigma_j"] \arrow[dd, "\delta_i"] \arrow[ddr, phantom, "c"] & \underset{I^{k+1}}{\mathrm{hocolim}}\, G_{k+1}\circ\delta_j \arrow[r, "\mathrm{can}"] \arrow[d, "\delta_i\circ\mathrm{id}"] \arrow[ddr, phantom, "b"] & \underset{I^{k+2}}{\mathrm{hocolim}}\, G_{k+1} \arrow[dd, "\delta_i"] \\
 & \underset{I^{k+1}}{\mathrm{hocolim}}\, G_k\circ\partial_i\circ\delta_j \arrow[dr, "\mathrm{can}"] \arrow[d, equal] & \\
\underset{I^{k+1}}{\mathrm{hocolim}}\, G_{k-1}\circ\partial_i \arrow[r, "\sigma_{j-1}\circ\mathrm{id}"] \arrow[d, "\mathrm{can}"] \arrow[dr, phantom, "b"] & \underset{I^{k+1}}{\mathrm{hocolim}}\, G_k\circ\delta_{j-1}\circ\partial_i \arrow[d, "\mathrm{can}"] \arrow[dr, phantom, "a"] & \underset{I^{k+2}}{\mathrm{hocolim}}\, G_{k+1}\circ\partial_i \arrow[d, "\mathrm{can}"] \\
\underset{I^{k}}{\mathrm{hocolim}}\, G_{k-1} \arrow[r, "\sigma_{j-1}"] & \underset{I^{k}}{\mathrm{hocolim}}\, G_k\circ\delta_{j-1} \arrow[r, "\mathrm{can}"] & \underset{I^{k+1}}{\mathrm{hocolim}}\, G_k
\end{tikzcd}$$

a : def'n of "can."

b : naturality of "can."

c : $(\delta_i\circ\mathrm{id})\sigma_j = (\sigma_{j-1}\circ\mathrm{id})\delta_i$. //

We next examine the homotopy type of

$$THH(E)_k := \underset{I^{k+1}}{\mathrm{hocolim}}\, G_k(E).$$

Let us first show that the index cat. is contractible. A functor $f: I \to I'$ induces a cts. map $|f|: |I| \to |I'|$, and a nat'l transformation $\alpha: f \to f'$ induces a homotopy $h_\alpha : |I| \times [0,1] \to |I'|$ from $|f|$ to $|f'|$. We consider the functors

$$pr_1, \amalg, pr_2 : I \times I \to I$$

and the natural transformations

$$\begin{array}{ccccc} \underline{i} & \xrightarrow{\iota} & \underline{i} \amalg \underline{i}' & \xleftarrow{\iota'} & \underline{i}' \\ s & \longmapsto & s & & \\ & & s + i & \longleftarrow\!\shortmid & s \end{array}.$$

It follows that we have homotopies

$$|pr_1| \simeq |\amalg| \simeq |pr_2| : |I| \times |I| \to |I|.$$

But then $|I|$ is contractible.

Similarly, the inclusion $j : I_{\geq N} \to I$ of the full subcategory on objects $\underline{i}$, $i \geq N$, is a htpy. eq. with homotopy inverse $r : I \to I_{\geq N}$, $r(\underline{i}) = \underline{i} \amalg \underline{N}$:

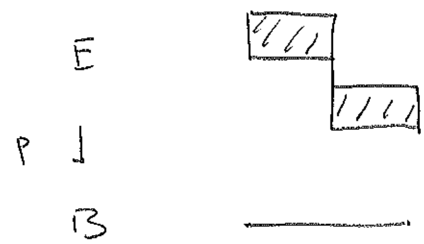
$$(j \circ r)(\underline{i}) = \underline{i} \amalg \underline{N} \xleftarrow{\iota} \underline{i}$$

$$(r \circ j)(\underline{i}) = \underline{i} \amalg \underline{N} \xleftarrow{\iota} \underline{i}.$$

A map $p: E \to B$ of unpointed spaces is a <u>quasi</u>-fi<u>bration</u> if for all $x \in B$, the canonical map from the fiber to the homotopy fiber over x is a w.e. This conditions implies a l.e.s. $(e \in p^{-1}(x))$

$$\cdots \to \pi_q(p^{-1}(x), e) \to \pi_q(E, e) \to \pi_q(B, x) \to \cdots,$$

A Serre fibration is a quasi-fibration but not conversely:



Quasi-fibrations "glue" [Dold-Thom]. The following result is the lemma in the proof of thm. B of Quillen, Higher algebraic K-theory I.

<u>Lemma</u> Let X be an I-diagram of unpointed spaces and suppose that for all morphisms $i \to i'$ in I, the map $X_i \to X_{i'}$ is a weak equivalence. Then the canonical map

$$\operatorname*{hocolim}_I X \longrightarrow |X|$$

is a quasi-fibration. //

The proof of loc. cit. shows more generally that if $X_i \to X_{i'}$ is s-connected, for all $i \to i'$ in I, then the can. map.

$$\operatorname{hocolim}_I X \xrightarrow{p} |X|$$

has the property that for all $x \in |X|$, the canonical map

$$\text{fiber over } x \longrightarrow \text{htpy fiber over } x$$

is s-connected.

<u>Def</u> (Bökstedt) A small category I is a good index category if it has the following properties.

(i) There is an assoc. and unital mult.

$$I \times I \xrightarrow{\mu} I.$$

(ii) There are natural transf. between μ and the projections pr_1 and pr_2.

(iii) There is a filtration

$$I = \mathrm{Fil}^0 I \supset \mathrm{Fil}^1 I \supset \cdots$$

s.t. $\mu(\mathrm{Fil}^s I, \mathrm{Fil}^t I) \subset \mathrm{Fil}^{s+t} I \neq \emptyset$. //

If I is a good index category then so is I^n, for all $n \geq 1$.

<u>Lemma</u> If I is a good index category and X an I-diagram of unpointed spaces then the inclusion

$$\operatorname*{hocolim}_{\mathrm{Fil}^s I} X \circ j \xrightarrow{\text{can}} \operatorname*{hocolim}_{I} X$$

is a deformation retract.

<u>proof</u> let $j : \mathrm{Fil}^s I \to I$ be the canonical inclusion. We pick an obj. c of $\mathrm{Fil}^s I$ and let $a : I \to \mathrm{Fil}^s I$ be the functor that takes i to $\mu(i, c)$. There is a natural transf.

$$\mathrm{id}_I \xrightarrow{\iota} j \circ a$$

...

$$
\begin{tikzcd}
I \arrow[d, "i_0"] \arrow[dr, equal] & \\
I \times \{0 \to 1\} \arrow[r, "h"] & I \\
I \arrow[u, "i_1"'] \arrow[ur, "a"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
h(\underline{i}, 0) \arrow[r, phantom, "="] \arrow[d] & \underline{i} \arrow[d, "\iota"] \\
h(\underline{i}, 1) \arrow[r, phantom, "="] & \underline{i} \amalg \underline{N}
\end{tikzcd}
$$

Natural transf. $\alpha : pr_1 \to h$:

$$
\begin{tikzcd}
pr_1(\underline{i}, 0) \arrow[r, phantom, "="] \arrow[d] & \underline{i} \arrow[r, "id"] \arrow[d, "id"] & \underline{i} \arrow[r, phantom, "="] \arrow[d, "\iota"] & h(\underline{i}, 0) \arrow[d] \\
pr_1(\underline{i}, 1) \arrow[r, phantom, "="] & \underline{i} \arrow[r, "\iota"] & \underline{i} \amalg \underline{N} \arrow[r, phantom, "="] & h(\underline{i}, 1)
\end{tikzcd}
$$

Given $X : I \to Top$, we get

$$
\begin{tikzcd}
\operatorname{hocolim}_I X \arrow[d, equal] \arrow[ddrr, bend left, "id"] & & \\
\operatorname{hocolim}_I (X \circ pr_1 \circ i_0) \arrow[d, "can"] & & \\
\operatorname{hocolim}_{I \times \{0 \to 1\}} (X \circ pr_1) \arrow[r, "\alpha"] & \operatorname{hocolim}_{I \times \{0,1\}} (X \circ h) \arrow[r, "can"] & \operatorname{hocolim}_I X \\
\operatorname{hocolim}_I (X \circ pr_1 \circ i_1) \arrow[u, "can"] \arrow[r, "\iota"] & \operatorname{hocolim}_I (X \circ h \circ i_1) \arrow[u, "can"] \arrow[d, equal] & \\
\operatorname{hocolim}_I X \arrow[u, equal] \arrow[r, "\iota"] & \operatorname{hocolim}_I (X \circ a) \arrow[uur, bend right, "can"'] &
\end{tikzcd}
$$

This shows that the composite

$$\operatorname*{hocolim}_{I} X \xrightarrow{c} \operatorname*{hocolim}_{I} (X \circ a) \xrightarrow{can} \operatorname*{hocolim}_{I} X$$

is homotopic to the identity. The same is true for the composite

$$\operatorname*{hocolim}_{I_{\geq N}} X \xrightarrow{c} \operatorname*{hocolim}_{I_{\geq N}} (X \circ a) \xrightarrow{can} \operatorname*{hocolim}_{I_{\geq N}} X$$

$$\begin{tikzcd}
 & \operatorname{hocolim}_{I} X \arrow[r, "\sim"] \arrow[rr, bend left, "\mathrm{id}"] \arrow[r, phantom, shift left=4, "\simeq"] \arrow[dr, dashed] & \operatorname{hocolim}_{I} X\circ j\circ a \arrow[r, "\mathrm{can}"] \arrow[d, "\mathrm{can}"] & \operatorname{hocolim}_{I} X \\
\operatorname{hocolim}_{I_{\geq N}} X\circ j \arrow[ur, "\mathrm{can}"] \arrow[r, "\sim"] \arrow[rr, bend right, "\mathrm{id}"'] \arrow[r, phantom, shift right=4, "\simeq"] & \operatorname{hocolim}_{I_{\geq N}} X\circ j\circ a\circ j \arrow[r, "\mathrm{can}"] & \operatorname{hocolim}_{I_{\geq N}} X\circ j \arrow[ur, "\mathrm{can}"] &
\end{tikzcd}$$

Since the map

$$\operatorname{hocolim}_{I_{\geq N}} X\circ j \longrightarrow \operatorname{hocolim}_{I} X$$

has the homotopy extension property, the diagram above shows that it is deformation retract.