3/9/04

Let $E$ be a symmetric ring spectrum and let $X$ be a pointed space. Let

$$I^{k+1} \xrightarrow{G_k(E;X)} Top$$

be the functor defined on objects by

$$G_k(E;X)(\underline{i_0}, \dots, \underline{i_k})$$

$$= F(S^{i_0} \wedge \dots \wedge S^{i_k}, E_{i_0} \wedge \dots \wedge E_{i_k} \wedge X).$$

Let $(n)$ be the finite ordered set

$$(n) = \{1, 2, \dots, n\}$$

and let $I^{(n)}$ be the product category. It is a strict monoidal category under component-wise concatenation of sets and maps $\sqcup^{(n)}$. Moreover, there is a functor

$$I^{(n)} \xrightarrow{\sqcup_n} I$$

$$(\underline{i_1}, \dots, \underline{i_n}) \longmapsto \underline{i_1} \sqcup \dots \sqcup \underline{i_n}$$

(It does not preserve the monoidal structure unless $n = 0$ or $1$.) Let

$$G_k^{(n)}(E;X) = G_k(E;X) \circ (\sqcup_n)^{k+1}$$

and define

$$THH^{(n)}(E;X)_k = \operatorname*{hocolim}_{(I^{(n)})^{k+1}} G_k^{(n)}(E;X).$$

As $k$ varies, these spaces form a cyclic space with structure maps defined as before. This gives the $\mathbb{T}$-space

$$THH^{(n)}(E;X) = |[k] \mapsto THH^{(n)}(E;X)_k|.$$

An ordered inclusion $(n) \to (n')$ gives rise to a map of cyclic spaces

$$THH^{(n)}(E;X)_k \to THH^{(n')}(E;X)_k.$$

It is an equivalence for $n \geq 1$ by the approximation lemma. For $n=0$, $I^{(0)}$ is the category with one object and one morphism, and $I^{(0)} \xrightarrow{\mu_0} I$ takes the unique object to $\underline{0}$. So there is a can. isom.

$$N_k^{cy}(E_0) \wedge X \xrightarrow{\sim} THH^{(0)}(E;X)$$

where $N_\cdot^{cy}(E_0)$ is the cyclic bar-constr. of the pointed monoid $(E_0, \mu_{0,0})$.

Symmetric spectrum $T(E)$ with $n$'th space

$$T(E)_n = THH^{(n)}(E; S^n) .$$

There are two $\Sigma_n$-actions. One is induced by the permutation action on $I^{(n)}$; the other from the action on $S^n$. We give $T(E)_n$ the diagonal $\Sigma_n$-action. Let

$$I^{(n)} \xrightarrow{\iota_{m,n}} I^{(m+n)}$$

be the inclusion as the last $n$ factors. Then the canonical maps define a natural transformation

$$S^m \wedge G_k^{(n)}(E; S^n) \xrightarrow{\alpha_{m,n}} G_k^{(m+n)}(E; S^{m+n}) \circ \iota_{m,n}$$

and hence we have a map

$$S^m \wedge \operatorname*{hocolim}_{(I^{(n)})^{k+1}} G_k^{(n)}(E; S^n)$$

$$\underset{\sim}{\xleftarrow{\text{can}}} \operatorname*{hocolim}_{(I^{(n)})^{k+1}} S^m \wedge G_k^{(n)}(E; S^n)$$

$$\xrightarrow{\alpha_{m,n}} \operatorname*{hocolim}_{(I^{(n)})^{k+1}} G_k^{(m+n)}(E; S^{m+n}) \circ \iota_{m,n}$$

$$\xrightarrow{\text{can}} \operatorname*{hocolim}_{(I^{(m+n)})^{k+1}} G_k^{(m+n)}(E; S^{m+n}) .$$

or

$$S^m \wedge THH^{(n)}(E; S^n)_k \longrightarrow THH^{(m+n)}(E; S^{m+n})_k .$$

As $k$ varies, these maps define a map of cyclic spaces, so we get

$$S^m \wedge T(E)_n \xrightarrow{\sigma_{m,n}} T(E)_{m+n} .$$

This makes $T(E)$ a symmetric spectrum.

There is an isomorphism of categories

$$(I^{(m)})^{k+1} \times (I^{(n)})^{k+1} \xrightarrow[\sim]{\tau_{m,n}} (I^{(m+n)})^{k+1}$$

which "twists" the factors as follows.

$$(((i_{0,1}, \dots, i_{0,m}), \dots, (i_{k,1}, \dots, i_{k,m})), ((i'_{0,1}, \dots, i'_{0,n}), \dots, (i'_{k,1}, \dots, i'_{k,n})))$$

$$\longmapsto ((i_{0,1}, \dots, i_{0,m}, i'_{0,1}, \dots, i'_{0,n}), \dots, (i_{k,1}, \dots, i_{k,m}, i'_{k,1}, \dots, i'_{k,n})).$$

The canonical maps and mult. in $E$ gives

$$G_k^{(m)}(E; S^m) \wedge G_k^{(n)}(E; S^n) \longrightarrow G_k^{(m+n)}(E; S^{m+n}) \circ \tau_{m,n}$$

and hence a map

$$\underset{(I^{(m)})^{k+1}}{\operatorname{hocolim}}\, G_k^{(m)}(E; S^m) \wedge \underset{(I^{(n)})^{k+1}}{\operatorname{hocolim}}\, G_k^{(n)}(E; S^n)$$

$$\underset{\sim}{\xleftarrow{\text{can}}} \operatorname*{hocolim}_{(I^{(m)})^{k+1} \times (I^{(n)})^{k+1}} G_k^{(m)}(E; S^m) \wedge G_k^{(n)}(E; S^n)$$

$$\longrightarrow \operatorname*{hocolim}_{(I^{(m)})^{k+1} \times (I^{(n)})^{k+1}} G_k^{(m+n)}(E; S^{m+n}) \circ \amalg_{m,n}$$

$$\underset{\sim}{\xrightarrow{\text{can}}} \operatorname*{hocolim}_{(I^{(m+n)})^{k+1}} G_k^{(m+n)}(E; S^{m+n})$$

or

$$THH^{(m)}(E; S^m)_k \wedge THH^{(n)}(E; S^n)_k$$

$$\longrightarrow THH^{(m+n)}(E; S^{m+n})_k .$$

If $E$ is commutative, these maps form a map of cyclic spaces, so we get

$$T(E)_m \wedge T(E)_n \xrightarrow{\mu_{m,n}} T(E)_{m+n} .$$

There is a map

$$S^n \xrightarrow{\eta_n} T(E)_n$$

which is defined in a manner similar

to the map $\sigma_{m,n}$. These maps make $T(E)$ a commutative symmetric ring spectrum.

$$H_q(E;k) = \operatorname{colim}_n \tilde{H}_{q+n}(E_n;k)$$

$$H_*(\mathbb{T};k) \otimes H_*(T(E);k) \to H_*(T(E);k)$$

$$[\mathbb{T}] \otimes x \longmapsto dx$$

Prop Let $E$ be a symmetric ring spectrum and let $k$ be a field. Then there is a spectral sequence

$$E^2 = HH_*(H_*(E;k)/k) \Rightarrow H_*(T(E);k)$$

which is multiplicative if $E$ is comm.

pf Assume that $E_n$ is well-pointed and $(n-1)$-connected. Recall

$$T(E)_n = |[k] \mapsto THH^{(n)}(E;S^n)[k]|$$

so we have a spectral sequence

$$E^2 = H_*(\tilde{H}_{*+n}(THH^{(n)}(E;S^n)[-];k), \Sigma(-1)^i d_i)$$

$$\Rightarrow \tilde{H}_{*+n}(T(E)_n;k),$$

and we obtain the spectral sequence by taking colimit over $n$. To evaluate $E^2$

By the approximation lemma,

$$\tilde{H}_{q+n}(THH^{(n)}(E;S^n)[k];k)$$

$$\xleftarrow{\sim} \tilde{H}_{q+n}(F(S^{i_0}\wedge\cdots\wedge S^{i_k}, E_{i_0}\wedge\cdots\wedge E_{i_k}\wedge S^n);k),$$

if $i_0,\ldots,i_k$ are sufficiently large. And since $E_m$ is $(m-1)$-connected, the homology suspension

$$\tilde{H}_{q+n}(F(S^{i_0}\wedge\cdots\wedge S^{i_k}, E_{i_0}\wedge\cdots\wedge E_{i_k}\wedge S^n);k)$$

$$\xrightarrow{\sim} \tilde{H}_{q+n+i}(E_{i_0}\wedge\cdots\wedge E_{i_k}\wedge S^n;k)$$

is an isom., if $n$ is sufficiently large ($i = i_0+\cdots+i_k$). Since $k$ is a field, this group is the degree $(q+i)$-part of

$$\underbrace{\tilde{H}_*(E_{i_0};k)\otimes\cdots\otimes\tilde{H}_*(E_{i_k};k)}_{k+1}.$$

But this group is canonically isom. to the degree $q$ part of

$$\underbrace{H_*(E;k)\otimes\cdots\otimes H_*(E;k)}_{k+1}$$

if $i_0,\ldots,i_k$ are sufficiently large. //

$$E^2 = H_*\left(\tilde H_{*+n}(\iota_*\iota^*\iota_* THH^{(n)}(E; S^n)[-]; k), \Sigma(-1)^i d_i\right)$$

$$\Rightarrow \tilde H_{*+n}(|\iota_*\iota^*\iota_* THH^{(n)}(E; S^n)[-]|; k)$$

$$|\iota_*\iota^*\iota_*(pt)|_+ \wedge |\iota_* X| \xrightarrow{\text{hard}} |\iota_*\iota^*\iota_*(pt)|_+ \wedge |\iota_* X|$$

$$\xleftarrow[\sim]{h} |\iota_*\iota^*\iota_* X| \xrightarrow{\varepsilon} |X|$$

All maps are filtration-preserving w.r.t. to the skeleton filtrations, so induce maps of the assoc. spectral sequences. The spectral sequence for

$$\tilde H_*(|\iota_*\iota^*\iota_*(pt)|_+ \wedge |\iota_* X|; k)$$

is the tensor product of the spectral sequences for $H_*(|\iota_*\iota^*\iota_*(pt)|; k)$ and $\tilde H_*(|\iota_* X|; k)$. The former takes the form


so there are no differentials for degree reasons. So $[\pi]$ survives to $E^\infty$

$E = \widetilde{\mathbb{F}}_p$ , $k = \mathbb{F}_p$

$H_*(E;k) = \mathcal{A}$ = dual Steenrod alg.

Suppose $p = 2$. Then

$$\mathcal{A} = S_{\mathbb{F}_2}\{\zeta_1, \zeta_2, \ldots\}, \quad \deg \zeta_i = 2^i - 1 .$$

Calc.

$$E^2 = HH_*(\mathcal{A}) = \mathrm{Tor}_*^{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}) .$$

Let

$$\mathcal{A}' = S_{\mathbb{F}_2}\{\zeta_1 \otimes 1 - 1 \otimes \zeta_1, \zeta_2 \otimes 1 - 1 \otimes \zeta_2, \ldots\} \subset \mathcal{A}^e .$$

Then

$$\mathcal{A} \otimes \mathcal{A}' \xrightarrow{\sim} \mathcal{A}^e$$

$$a \otimes x \longmapsto (a \otimes 1) \cdot x$$

so

$$\mathrm{Tor}_*^{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}) \xleftarrow{\sim} \mathrm{Tor}_*^{\mathcal{A} \otimes \mathcal{A}'}(\mathcal{A} \otimes \mathbb{F}_2, \mathcal{A} \otimes \mathbb{F}_2)$$

$$\xleftarrow{\sim} \mathrm{Tor}_*^{\mathcal{A}}(\mathcal{A}, \mathcal{A}) \otimes \mathrm{Tor}_*^{\mathcal{A}'}(\mathbb{F}_2, \mathbb{F}_2)$$

$$= \mathcal{A} \otimes \mathrm{Tor}_*^{\mathcal{A}'}(\mathbb{F}_2, \mathbb{F}_2) .$$

$$\xleftarrow{\sim} \mathcal{A} \otimes \mathrm{Tor}_*^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$$

$$= \mathcal{A} \otimes \Lambda_{\mathbb{F}_2}\{\sigma\zeta_1, \sigma\zeta_2, \ldots\}.$$

So as a bi-graded $\mathbb{F}_2$-algebra

$$E^2 = S_{\mathbb{F}_2}\{\zeta_1, \zeta_2, \ldots\} \otimes \Lambda_{\mathbb{F}_2}\{\sigma\zeta_1, \sigma\zeta_2, \ldots\}$$

$$\deg \zeta_i = (0, 2^i - 1)$$

$$\deg(\sigma\zeta_i) = (1, 2^i - 1).$$

Since the differentials are derivations and since all the algebra gen. are contracted on the lines $s=0$ and $s=1$, the diff. must all be zero. Hence $E^\infty = E^2$ as a bi-graded algebra. It follows that as a bigraded algebra

$$\mathrm{gr.}\, H_*(T(\mathbb{F}_2); \mathbb{F}_2) = \mathcal{A} \otimes \Lambda_{\mathbb{F}_2}\{\sigma\zeta_1, \sigma\zeta_2, \ldots\}.$$

However, there are multiplicative extensions in passing from $\mathrm{gr.}\, H_*(T(\mathbb{F}_2); \mathbb{F}_2)$ to

$H_*(T(\mathbb{F}_2);\mathbb{F}_2)$. In the latter, it turns out, $(\sigma\xi_i)^2 = \sigma\xi_{i+1}$ s.t. as a graded $\mathbb{F}_2$-algebra

$$H_*(T(\mathbb{F}_2),\mathbb{F}_2) = \mathcal{A} \otimes S_{\mathbb{F}_2}\{\sigma\},$$

where $\sigma$ is represented by $\sigma\xi_1$ in the spectral sequence.

In general, if $E$ is a commutative symmetric ring spectrum, then (in the homotopy category), $T(E)$ is an $E$-algebra spectrum, in particular, an $E$-module spectrum. So $T(\mathbb{F}_2)$ is an $\tilde{\mathbb{F}}_2$-module spectrum. Every such spectrum is a wedge of (suspended) copies of $\tilde{\mathbb{F}}_2$. It follows that

$$T_*(\mathbb{F}_2) = S_{\mathbb{F}_2}\{\sigma\}.$$

Geometric argument by Cohen - Schlichtkrull:

$X \xrightarrow{f} BO \rightsquigarrow$

$M(f)$ = Thom (orthogonal) spectrum:

Choose a filtration

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X$$

s.t. $f(X_n) \subset BO(n)$, and consider the pull-back bundle

$$\begin{array}{ccc} E_n & \longrightarrow & EO(n) \times_{O(n)} \mathbb{R}^n \\ \downarrow & & \downarrow \\ X_n & \xrightarrow{f_n} & BO(n) \end{array}$$

Then

$$M(f)_n = Th(E_n \downarrow X_n) = D(E_n) / S(E_n).$$

If $f$ can be de-looped

$$\begin{array}{ccc} X & \xrightarrow{f} & BO \\ \downarrow & & \downarrow \\ \Omega D X & \xrightarrow{\Omega B f} & \Omega B B O \end{array}$$

then $M(f)$ can be made an orthogonal ring spectrum.

Thm (Cohen-Schlichtkrull) If $f$ is a double loop map, then there is a natural equivalence

$$T(M(f)) \simeq M(f) \wedge M(BX \xrightarrow{Bf} BBO \xrightarrow{\eta} BO). \;//$$

Let $\eta: S^1 \to BO$ represent a generator of $\pi_1 BO = \{\pm 1\}$ and extend $\eta$ to

$$\Omega^2 S^3 \xrightarrow{f} BO.$$

By Mahowald, $M(f) \simeq \tilde{\mathbb{F}}_2$, so the theorem gives

$$T(\tilde{\mathbb{F}}_2) \simeq \tilde{\mathbb{F}}_2 \wedge M(\Omega S^3 \to BO)$$

$$\simeq \tilde{\mathbb{F}}_2 \wedge \Omega S^3_+$$

Thom iso.

So $T_*(\tilde{\mathbb{F}}_2) \cong S_{\mathbb{F}_2}\{\sigma\}$, $\deg \sigma = 2$.

 3/30/04

$G$ = cpt. Lie group.

$J_G$ = top'd $G$-category

= category enriched in the symm. monoidal cat. of pointed $G$-spaces and smash product.

obj. : all f.-d. orthogonal $G$-repr.

morph. : let $O(V,V')$ be the $G$-space of linear isometries from $V$ to $V'$ with $G$ acting by conjugation; let

$$\begin{array}{ccc} E(V,V') & \hookrightarrow & O(V,V') \times V' \\ \downarrow & & \downarrow \\ O(V,V') & = & O(V,V') \end{array}$$

be the subbdl. of pairs $(f, x)$ with $x \in V' \setminus f(V)$. It is a $G$-bdl. and

$$\underline{\mathrm{Hom}}_{J_G}(V,V') = Th(E(V,V'))$$

is the Thom-G-space.

comp: Is the G-map

$$\underline{\mathrm{Hom}}_{J_G}(V',V'') \wedge \underline{\mathrm{Hom}}_{J_G}(V,V')$$

$$\xrightarrow{\circ} \underline{\mathrm{Hom}}_{J_G}(V,V'')$$

$$((g,y),(f,x))$$

$$\longmapsto (g\circ f,\ g(x)+y).$$

An <u>orthogonal</u> G-<u>spectrum</u> is a continuous G-equivariant functor

$$J_G \xrightarrow{E} \mathrm{Top}_G.$$

$\qquad\qquad\qquad\qquad\qquad\qquad \|$

pointed G-spaces
and non-equivariant
pointed maps

Two special cases:

(1) $\dim V = \dim V'$ :

$$\underline{\mathrm{Hom}}_{\mathcal{J}_G}(V, V') = O(V, V')_+$$

and that $E$ is a functor implies that there is a can. homeo.

$$O(V, V')_+ \underset{O(V)}{\wedge} E(V') \xrightarrow{\sim} E(V')$$

that is $G \times O(V)$ - equivariant.

(2) $V \subset V'$ : let $V' - V \subset V'$ be the orthogonal complement; then

$$\underline{\mathrm{Hom}}_{\mathcal{J}_G}(V, V') = O(V')_+ \underset{O(V'-V)}{\wedge} S^{V'-V}$$

and that $E$ is a functor implies that there is a can. map

$$S^{V'-V} \wedge E(V) \longrightarrow E(V')$$

that is $G \times O(V'-V) \times O(V)$ - equivariant.

Let $E$ and $E'$ be orthogonal $G$-spectra.

$\underline{\mathrm{Hom}}(E, E')$ = pointed $G$-space of natural transf. from $E$ to $E'$ considered as continuous functors.

$\underline{\mathrm{Hom}}_G(E, E')$ = pointed space of natural transf. from $E$ to $E'$ considered as cts. $G$-functors.

Then

$$\underline{\mathrm{Hom}}_G(E, E') = \underline{\mathrm{Hom}}(E, E')^G .$$

Adjoint pair of $G$-functors

$$(\mathrm{Top}_G)^{\mathcal{J}_G} \underset{F_V}{\overset{ev_V}{\rightleftarrows}} \mathrm{Top}_G$$

($G$-spaces of maps).

level model structure: The category of orthogonal $G$-spectra and $G$-maps has a cofibrantly generated model str.:

weak equivalences = obj.-wise w.e. of $G$-spaces.

fibrations = obj.-wise Serre fibration of $G$-spaces,

cofibrations = LLP w.r.t. trivial fibr.

Generators:

$$FI: \quad F_V((G/H \times S^{n-1})_+) \to F_V((G/H \times D^n)_+)$$

$$FJ: \quad F_V((G/H \times D^n)_+) \to F_V((G/H \times D^n \times [0,1])_+)$$

when $V$ runs through a set of repr. of iso. classes of all f.-d. orthogonal $G$-repr. and $H \subset G$ through all closed subgroups.

Obtain stable model str. by localizing w.r.t. to the following set of maps.

$$A = \left\{ F_{V\oplus W} S^W \xrightarrow{\lambda_{V,W}} F_V S^0 \right\}.$$

The map $\lambda_{V,W}$ is the adjoint of the map of pointed G-spaces

$$S^W \longrightarrow (F_V S^0)(V\oplus W) \underset{\text{can.}}{\cong} O(V\oplus W)_+ \wedge_{O(W)} S^W$$

$$w \longmapsto (e, w)$$

An orthogonal G-spectrum $E$ is A-local if the induced maps

$$[F_V S^0, E)^{\text{level}} \xrightarrow{\lambda_{V,W}^*} [F_{V\oplus W} S^W, E]^{\text{level}}$$

are bijections or, equivalently, if the adjoint str. maps

$$E(V) \longrightarrow \Omega^{W-V} E(W)$$

are w.e. of pointed G-spaces, $V \subset W$.

A map $X \xrightarrow{f} X'$ of orthogonal $G$-spectra is an $A$-local w.e. if for all $A$-local $E$,

$$[X', E]^{\text{level}} \xrightarrow{f^*} [X, E]^{\text{level}}$$

is a bijection.

Stable model str.: The category of orthogonal $G$-spectra and $G$-maps has a cofibrantly gen. model str.:

weak equivalences : $A$-local w.e.
cofibrations : level cofibrations
fibrations : RLP w.r.t. trivial cof.

Generators:

$$FI : F_V((G/H \times S^{n-1})_+) \to F_V((G/H \times D^n)_+)$$

$$K : F_V((G/H \times D^n)_+) \to F_V((G/H \times D^n \times [0,1])_+)$$

and the maps $(i_m, k_{V,W})$ where

$$F_{V \oplus W} S^W \xrightarrow{k_{V,W}} \mathrm{Cyl}(\lambda_{V,W}).$$

Define homotopy groups

$$\pi_q^H(X) = \operatorname{colim}_V \pi_q^H(\Omega^V X(V)).$$

Prop A map $X \xrightarrow{f} X'$ of orthogonal $G$-spectra is a stable equivalence if and only if $\pi_q^H(f)$ is an isom. for all integers $q$ and all $H \subset G$. //

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If $E$ is a symmetric ring spectrum we get an orthogonal $\mathbb{T}$-spectrum $T(E)$ with $V$'th space

$$T(E)(V) = THH(E; S^V).$$

There are two $\mathbb{T}$-actions: One comes from the cyclic structure of $THH(E)[-]$; the other is induced by the $\mathbb{T}$-action on $S^V$. We give $T(E)(V)$ the diagonal $\mathbb{T}$-action.

Let $G$ be a cpt. Lie grp. A pointed $G$-CW-complex is a pointed (left) $G$-space $X$ together with a sequence of pointed sub-$G$-spaces

$$* = sk_{-1}X \subset sk_0 X \subset sk_1 X \subset \cdots \subset X$$

and for all $n \geq 0$, a push-out square of un-pointed $G$-spaces

$$\begin{tikzcd} \coprod_\alpha G/H_\alpha \times S^{n-1} \arrow[r, "\varphi_n"] \arrow[d] & sk_{n-1}X \arrow[d, hook] \\ \coprod_\alpha G/H_\alpha \times D^n \arrow[r, "\varphi_n"] & sk_n X \end{tikzcd}$$

s.t. the canonical pointed $G$-map

$$\operatorname{colim}_n sk_n X \longrightarrow X$$

is a homeomorphism.

A pointed $G$-map $f: X \to X'$ between pointed $G$-CW-complexes is cellular if $f(sk_n X) \subset sk_n X'$ for all $n \geq -1$.

Lemma Let $G$ be a cpt. Lie group, let $X$ be a pointed $G$-CW-complex, and let $d(H)$ be the supremum of the dimension of the cells of $X$ of orbit-type $G/H$. Let $Y$ be a pointed $G$-space such that $Y^H$ is $n(H)$-conn. Then the equivariant mapping space

$$F_G(X,Y) = F(X,Y)^G$$

is $m$-connected with

$$m = \inf \{ n(H) - d(H) \mid H \in \mathcal{O}_G(X) \} .$$

Here $\mathcal{O}_G(X)$ is the set of subgroups $H \subset G$ s.t. $X$ has a cell of orbit-type $G/H$.

Proof Show by induction on $n$ that $F(\mathrm{sk}_n X, Y)^G$ is $m$-connected. The case $n = -1$ is trivial, so assume $F(\mathrm{sk}_{n-1} X, Y)^G$ is $m$-connected. The map $\mathrm{sk}_{n-1} X \to \mathrm{sk}_n X$ has the $G$-

homotopy extension property, and so the induced map

$$F(sk_n X, Y)^G \longrightarrow F(sk_{n-1} X, Y)^G$$

has the homotopy lifting property. The fiber of the latter over the base-point is can. homeo. to

$$F(sk_n X / sk_{n-1} X, Y)^G$$

$$\underset{\sim}{\xrightarrow{\Phi_n}} F(\bigvee_\alpha G/H_{\alpha +} \wedge S^n, Y)^G$$

$$\xrightarrow{\sim} \prod_\alpha F(G/H_{\alpha +} \wedge S^n, Y)^G$$

$$\xleftarrow{\sim} \prod_\alpha F(S^n, Y^{H_\alpha}).$$

Now $F(S^n, Y^{H_\alpha})$ is $(n(H_\alpha) - n)$-conn. and $n(H_\alpha) - n \geq m$. So the induction step follows. If $sk_n X = X$ for some $n$, we are done. In general

$$F(X, Y)^G \xrightarrow{\sim} \operatorname{holim}_n F(sk_n X, Y)^G. \qquad //$$

The functor $(-)^H$ preserves push-out diagrams in which one map is a closed inclusion. It also preserves sequential colimits along inclusions. Hence $X^H$ is built of cells of the form $(G/K \times D^n)^H$. Since

$$D^n \subset (G/H \times D^n)^H$$

we see that

$$d(H) \leq \dim(X^H) .$$

<u>Lemma</u> Let $G$ be a finite group, and let $f : X \to Y$ be a map of orthogonal $G$-spectra. Suppose that for all f.d. orthogonal $G$-repr. $\lambda$, and all subgroups $H \subset G$, the induced map

$$f_\lambda^H : X_\lambda^H \longrightarrow Y_\lambda^H$$

is $(\dim_{\mathbb{R}}(\lambda^H) + \varepsilon_H(\lambda))$-connected, where $\varepsilon_H(\lambda)$ tends to infinity with $\lambda$. Then $f$ is a $\pi_*$-isom.

Proof If $F$ is the (level-wise) homotopy fiber of $f$, then there is a l.e.s.

$$\cdots \to \pi_q^H(F) \to \pi_q^H(X) \xrightarrow{f_*} \pi_q^H(Y) \to \pi_{q-1}^H(F) \to \cdots$$

so it suffices to show that for all subgroups $H \subset G$ and all integers $q$, $\pi_q^H(F) = 0$. If $q \geq 0$, then

$$\pi_q^H(F) = \operatorname*{colim}_{\lambda} \pi_q(F(S^\lambda, F_\lambda)^H)$$

By the previous lemma, $F(S^\lambda, F_\lambda)^H$ is $m$-connected with

$$m = \inf\{n(K) - d(K) \mid K \in \mathcal{O}_H(F_\lambda)\}$$

$$\leq \inf\{\dim(\lambda^K) + \varepsilon_K(\lambda) - \dim(\lambda^K) \mid K \subset H\}$$

$$= \inf\{\varepsilon_K(\lambda) \mid K \subset H\}.$$

Our assumptions are that this number tends to infinity as $\lambda$ runs through all f.d. ortho. $G$-repr. Hence, $\pi_q^H(F_\lambda) = 0$. //

If $E$ is a symmetric ring spectrum we get an orthogonal $\mathbb{T}$-spectrum $T(E)$ with $\lambda$'th space

$$T(E)_\lambda = THH(E; S^\lambda).$$

There are two $\mathbb{T}$-actions on this space. One comes from the cyclic structure of $THH(E; S^\lambda)[-]$; the other comes from the $\mathbb{T}$-action on $S^\lambda$. We give $T(E)_\lambda$ the diagonal $\mathbb{T}$-action. The $O(\lambda)$-action on $T(E)_\lambda$ is induced from the $O(\lambda)$-action on $S^\lambda$.

Actually, it is better to define $T(E)$ to be the symmetric orthogonal $\mathbb{T}$-spectrum with $(n, \lambda)$'th space

$$T(E)_{n,\lambda} = THH^{(n)}(E; S^n \wedge S^\lambda)$$

with the diagonal $\mathbb{T}$- and $\Sigma_n$-actions and with the $O(\lambda)$-action induced from the $O(\lambda)$-action on $S^\lambda$. Then for $E$ commutative, there is a product map

$$T(E) \wedge T(E) \xrightarrow{\mu} T(E)$$

that makes $T(E)$ a symmetric orthogonal ring $\mathbb{T}$-spectrum.

Let $A$ be a commutative ring, and let $\Pi$ be a commutative pointed monoid. Define $\phi : A \to A(\Pi)$ by $\phi(a) = a \cdot 1$ and $\iota : \Pi \to A(\Pi)$ by $\iota(\pi) = 1 \cdot \pi$.

Prop The composite map

$$T(A) \wedge N^{cy}(\Pi) \xrightarrow{\phi \wedge \iota} T(A(\Pi)) \wedge N^{cy}(A(\Pi))$$

$$\xrightarrow{\mu} T(A(\Pi))$$

is an $\mathcal{F}$-equivalence.

$\ulcorner$ A map $f : X \to Y$ is an $\mathcal{F}$-equivalence if $\pi_*^H(f)$ is an isom. for all finite subgroups $H \subset G$. $\lrcorner$

Proof By the second lemma, we must show that for all finite subgroups $C \subset \Pi$, all non-neg. integers $n$, and all f-d. ortho. $\Pi$-repr. $\lambda$, the induced map

$$(T(A)_{n,\lambda})^C \wedge N^{us}(\Pi)^C \xrightarrow{(f_{n,\lambda})^C} (T(A(\Pi))_{n,\lambda})^C$$

is $(n + \dim(\lambda^C) + \varepsilon_C(\lambda))$ - connected, where $\varepsilon_C(\lambda)$ tends to infinity with $\lambda$. A $C$-equivariant isometric isom. $\lambda \xrightarrow{\sim} \lambda'$ induces a $C$-equivariant homeo. $T(A)_{n,\lambda} \xrightarrow{\sim} T(A)_{n,\lambda'}$. Hence, we may assume that

$$\lambda = m\rho_C = \underbrace{\rho_C \oplus \cdots \oplus \rho_C}_{m}$$

is a direct sum of copies of the regular repr. $\rho_C = \mathbb{R}[C]$. We will show that the map

$$(f_{n,m\rho_C})^C$$

is $(n + 2m - 1)$ - connected.

4/6/04

We first spell out the maps

$$T(\tilde{A})_{n,\lambda} \wedge N^{cy}(\Pi) \xrightarrow{f_{n,\lambda}} T(\widetilde{A(\Pi)})_{n,\lambda}$$

The following composition of canonical maps

$$F(S^{i_0} \wedge \cdots \wedge S^{i_k}, \tilde{A}_{i_0} \wedge \cdots \wedge \tilde{A}_{i_k} \wedge S^{n+mpc}) \wedge \Pi^{\wedge(k+1)}$$

$$\to F(S^{i_0} \wedge \cdots \wedge S^{i_k}, \tilde{A}_{i_0} \wedge \cdots \wedge \tilde{A}_{i_k} \wedge \Pi^{\wedge(k+1)} \wedge S^{n+mpc})$$

$$\to F(S^{i_0} \wedge \cdots \wedge S^{i_k}, \widetilde{A(\Pi)}_{i_0} \wedge \cdots \wedge \widetilde{A(\Pi)}_{i_k} \wedge S^{n+mpc})$$

is a nat'l transf. of functors

$$G_k(\tilde{A}; S^{n+\lambda}) \wedge N^{cy}(\Pi)[k]$$

$$\to G_k(\widetilde{A(\Pi)}; S^{n+\lambda}).$$

This induces a nat'l transf.

$$G_k^{(n)}(\tilde{A}; S^{n+\lambda}) \wedge N^{cy}(\Pi)[k]$$

$$\to G_k^{(n)}(\widetilde{A(\Pi)}; S^{n+\lambda})$$

Take hocolim over $(I^{(n)})^{k+1}$ to get a map of pointed spaces

$$THH^{(n)}(\tilde{A}; S^{n+\lambda})[k] \wedge N^{cy}(\Pi)[k]$$

$$\to THH^{(n)}(\widetilde{A(\Pi)}; S^{n+\lambda})[k].$$

These maps constitute a map of cyclic spaces (i.e. they commute with the cyclic structure maps) so we get upon realization a map

$$THH^{(n)}(\tilde{A}; S^{n+\lambda}) \wedge N^{cy}(\Pi)$$

$$\to THH^{(n)}(\widetilde{A(\Pi)}; S^{n+\lambda}).$$

This is the map $f_{n,\lambda}$.

Let $\lambda = m\rho_C$. We next describe the induced map of $C$-fixed points

$$(T(\tilde{A})_{n,m\rho_C})^C \wedge N^{cy}(\Pi)^C$$

$$\to (T(\widetilde{A(\Pi)})_{n,m\rho_C})^C.$$

Let $r$ be the order of $C$ and recall that for a cyclic space $X[-]$, we have a canonical homeo.

$$|(sd_r X)[-]| \xrightarrow[\sim]{D_r} |X[-]|$$

and that on the left the action by the gen. $e^{2\pi i/r}$ of $C$ is induced from the simplicial map

$$\begin{array}{ccc} (sd_r X)[k] & = & X[r(k+1)-1] \\ \Big\downarrow & & \Big\downarrow (t_{r(k+1)-1})^{k+1} \\ (sd_r X)[-] & = & X[r(k+1)-1] \end{array}.$$

In the case at hand we consider

$$(sd_r THH^{(n)}(\tilde{A}; S^{n+m}\rho_C))[k]$$

$$= THH^{(n)}(\tilde{A}, S^{n+m}\rho_C)[r(k+1)-1]$$

with the diagonal $C$-action induced from a $C\times C$-action. The generator

$e^{2\pi i/r}$ of the first $C$-factor acts as $(t_{r(k+1)-1})^{k+1}$. The action of the second $C$-factor is induced from the $C$-action on $\rho_C$.

$$(\mathrm{sd}_r\, THH^{(n)}(\tilde{A}; S^{n+m\rho_C}))[k]$$

$$= \operatorname*{hocolim}_{(I^{(n)})^{r(k+1)}} G^{(n)}_{r(k+1)-1}(\tilde{A}; S^{n+m\rho_C}).$$

The action of the second $C$-factor is induced from a $C$-action on the individual terms of the htpy. colimit. But the action of the first $C$-factor is not: It also acts on the indexing category. However, let

$$(I^{(n)})^{k+1} \xrightarrow{\Delta_{r,k}} (I^{(n)})^{r(k+1)}$$

$$(i_0, \dots, i_k) \longmapsto (i_0, \dots, i_k, \dots, i_0, \dots, i_k)$$

and consider the canonical map

$$\operatorname*{hocolim}_{(I^{(n)})^{k+1}} G^{(n)}_{r(k+1)-1}(\bar{A}; S^{n+mpc}) \circ \Delta_{r,k}$$

$$\xrightarrow{can} \operatorname*{hocolim}_{(I^{(n)})^{r(k+1)}} G^{(n)}_{r(k+1)-1}(\bar{A}; S^{n+mpc}).$$

Then, on the domain, the actions of both $C$-factors are induced by $C$-actions on the individual terms of the homotopy colimit, and I claim that the induced map of (diagonal) $C$-fixed sets

$$\Big(\operatorname*{hocolim}_{(I^{(n)})^{k+1}} G^{(n)}_{r(k+1)-1}(\tilde{A}; S^{n+mpc}) \circ \Delta_{r,k}\Big)^C$$

$$\xrightarrow{\sim} \Big(\operatorname*{hocolim}_{(I^{(n)})^{r(k+1)}} G^{(n)}_{r(k+1)-1}(\tilde{A}; S^{n+mpc})\Big)^C$$

is a homeomorphism. The claim follows from the construction of homotopy colimits and from the fact that taking C-fixed sets preserves push-outs along closed inclusions and sequential colimits along inclusions. Similarly,

$$\underset{(I^{(n)})^{k+1}}{\mathrm{hocolim}} \left(G^{(n)}_{r(k+1)-1}(\tilde{A}; S^{n+mpc}) \circ D_{r,k}\right)^{C}$$

$$\xrightarrow{\sim} \left(\underset{(I^{(n)})^{k+1}}{\mathrm{hocolim}}\, G^{(n)}_{r(k+1)-1}(\tilde{A}; S^{n+mpc}) \circ D_{r,k}\right)^{C}$$

since the C-action is term-wise. So we have a can. homeo.

$$\left(sd_r\, THH^{(n)}(\tilde{A}; S^{n+mpc})[k]\right)^{C}$$

$$\xleftarrow{\sim} \underset{(I^{(n)})^{k+1}}{\mathrm{hocolim}} \left(G^{(n)}_{r(k+1)-1}(\tilde{A}; S^{n+mpc}) \circ D_{r,k}\right)^{C}.$$

A typical term in the homotopy colimit is canonically homeo. to the equivariant function space

$$F((S^{j_0}\wedge\cdots\wedge S^{j_k})^{\wedge r}, (\tilde{A}_{j_0}\wedge\cdots\wedge\tilde{A}_{j_k})^{\wedge r}\wedge S^{n+m\rho_C})^C.$$

Here $C$ acts on $(\cdots)^{\wedge r}$ by cyclic perm. of the $r$ smash factors and on $S^{n+m\rho_C}$ by the action induced from the $C$-action on $\rho_C$. The can. maps

$$F((S^{j_0}\wedge\cdots\wedge S^{j_k})^{\wedge r}, (\tilde{A}_{j_0}\wedge\cdots\wedge\tilde{A}_{j_k})^{\wedge r}\wedge S^{n+m\rho_C})\wedge(\Pi^{\wedge(k+1)})^{\wedge r}$$

$$(*)\quad \to F((S^{j_0}\wedge\cdots\wedge S^{j_k})^{\wedge r}, (\tilde{A}_{j_0}\wedge\cdots\wedge\tilde{A}_{j_k}\wedge\Pi^{\wedge(k+1)})^{\wedge r}\wedge S^{n+m\rho_C})$$

$$\to F((S^{j_0}\wedge\cdots\wedge S^{j_k})^{\wedge r}, (\widetilde{A(\Pi)}_{j_0}\wedge\cdots\wedge\widetilde{A(\Pi)}_{j_k})^{\wedge r}\wedge S^{n+m\rho_C})$$

are $C$-equivariant and define a nat'l transf.

$$(G^{(n)}_{r(k+1)-1}(\tilde{A}; S^{n+m\rho_C})\wedge N^{cy}_{r(k+1)-1}(\Pi))\circ\Delta_{r,k}$$

$$\to G^{(n)}_{r(k+1)-1}(\widetilde{A(\Pi)}; S^{n+m\rho_C})\circ\Delta_{r,k}.$$

Taking $C$-fixed sets and homotopy colimits over $(I^{(n)})^{k+1}$, we get a map

$$\left(sd_r\left(THH^{(n)}(\tilde{A}; S^{n+mpc}) \wedge N^{cy}_{\cdot}(\pi)\right)[k]\right)^C$$

$$\longrightarrow \left(sd_r THH^{(n)}(\widetilde{A(\pi)}; S^{n+mpc})[k]\right)^C$$

These maps constitute a map of simplicial space. The induced map of realizations together with the homeomorphism $D_r$ give a map

$$\left(THH^{(n)}(\tilde{A}; S^{n+mpc}) \wedge N^{cy}(\pi)\right)^C$$

$$\longrightarrow THH^{(n)}(\widetilde{A(\pi)}; S^{n+mpc})^C .$$

This is the map $(f_{n,mpc})^C$.

We must show that this map is $(n+2m-1)$-connected. It is induced by a map of simpl. spaces and it sufffices to show that the map of $k$-simplices is $(n+2m-1)$-conn. for all $k$. The spaces of $k$-simpl. are htpy. colimits over $(I^{(n)})^{k+1}$ and, by the approximation lemma, it sufffices to show that the maps of $C$-fixed points induced from the maps in $(*)$ are $(n+2m-1)$-connected for $j_0, \dots, j_k$ large. Write $j = j_0 + \dots + j_k$. We first consider the second map in $(*)$.

The map

$$\tilde{A}_i \wedge \Pi \longrightarrow \widetilde{A(\Pi)}_i$$

is $(2i-1)$-connected. Indeed, this is the inclusion of a wedge of copies of $\tilde{A}_i = K(A, i)$ indexed by $\Pi \setminus \{0\}$ in the corresponding (weak) product. It follows that the can. map

$$(\tilde{A}_{j_0} \wedge \dots \wedge \tilde{A}_{j_k} \wedge \Pi^{\wedge(k+1)})^{\wedge r} \wedge S^{n+m\rho_C}$$

$$\longrightarrow (\widetilde{A(\Pi)}_{j_0} \wedge \dots \wedge A(\Pi)_{j_k})^{\wedge r} \wedge S^{n+m\rho_C}$$

is $(2ir+n+mr-1)$-connected. Let $C_s \subset C_r$ and $r = st$. Then the induced map of $C_s$-fixed points is $n(C_s)$-conn. with $n(C_s) = 2it + n + mt - 1$. The dim. of

$$((S^{j_0} \wedge \cdots \wedge S^{j_k})^{\wedge r})^{C_s}$$

$$= (S^{j_0} \wedge \cdots \wedge S^{j_k})^{\wedge t}$$

is $jt$, so $d(C_s) \geq jt$. Hence by the lemma from last time the map of $C$-fixed sets induced by the second map in $(*)$ is $n$-conn. with

$$n = \min\{n(C_s) - d(C_s) \mid C_s \subset C_r\}$$

$$= j + n + m - 1.$$

This is $(n+2m-1)$-conn. if $j \geq m$.

We consider the first map in $(*)$. We write

$$X = S^{j_0} \wedge \cdots \wedge S^{j_k} \qquad Y = \tilde{A}_{j_0} \wedge \cdots \wedge \tilde{A}_{j_k}$$

$$Z = S^{n+mpc} \qquad P = \Pi^{\wedge(k+1)}$$

and consider the following diagram.

$$\begin{tikzcd}
F(X^{\wedge r}, Y^{\wedge r} \wedge Z) \wedge P^{\wedge r} \arrow[r] \arrow[d, "\tilde{\delta}"] & F(X^{\wedge r}, Y^{\wedge r} \wedge Z \wedge P^{\wedge r}) \arrow[d, "\tilde{\delta}_*"] \\
F(P^{\wedge r}, F(X^{\wedge r}, Y^{\wedge r} \wedge Z)) \arrow[r, "\cong"] & F(X^{\wedge r}, F(P^{\wedge r}, Y^{\wedge r} \wedge Z))
\end{tikzcd}$$

where $\tilde{\delta}$ is the adjoint of the pointed Kronecker $\delta$-function

$$P^{\wedge r} \wedge P^{\wedge r} \xrightarrow{\delta} S^0 .$$

The top horizontal map is the first map in $(*)$. We wish to show that the induced map of $C$-fixed points is $(n+2m-1)$-conn. We show that the map of $C$-fixed points induced by the left-hand vertical map is $(n+2m-1)$-conn. Consider

$$\begin{tikzcd}
F(X^{\wedge r}, Y^{\wedge r} \wedge Z)^C \wedge P \arrow[r, "\tilde{\delta}^C"] \arrow[d, "\tilde{\delta}"] & F(P^{\wedge r}, F(X^{\wedge r}, Y^{\wedge r} \wedge Z))^C \arrow[d, "\Delta^*"] \\
F(P, F(X^{\wedge r}, Y^{\wedge r} \wedge Z)^C) \arrow[r, "\cong"] & F(P, F(X^{\wedge r}, Y^{\wedge r} \wedge Z))^C
\end{tikzcd}$$

Here $F(X^{\wedge r}, Y^{\wedge r} \wedge Z)^C$ is $(n+m-1)$-conn., so the left-hand vertical map is $(2(n+m)-1)$-connected. The fiber of the right-hand vertical map is

$$F(P^{\wedge r}/P, F(X^{\wedge r}, Y^{\wedge r} \wedge Z))^C.$$

Since $P^{\wedge r}/P$ has no cells of orbit-type $C/C$, this space is $(n+tm-1)$-conn., where $t$ is the smallest non-trivial divisor in $r$. Since $t \geq 2$, we are done.

//

We define a natural $\mathbb{T}$-equivariant map

$$\rho_p^*\left(T(A)_{n,\lambda}\right)^{C_p} \xrightarrow{r_p} T(A)_{n,\rho_p^*\lambda^{C_p}}$$

and show that the induced map of $C_{p^{s-1}}$-fixed sets

$$\left(T(A)_{n,\lambda}\right)^{C_{p^s}} \longrightarrow \left(T(A)_{n,\rho_p^*\lambda^{C_p}}\right)^{C_{p^{s-1}}}$$

is $(n+\dim(\lambda)-1)$-connected. We have

$$T(A)_{n,\lambda} := THH^{(n)}(A; S^{n+\lambda})$$

$$= \left| THH^{(n)}(A; S^{n+\lambda})[-] \right|$$

$$\underset{\sim}{\xleftarrow{D}} \left| sd_{p^s} THH^{(n)}(A; S^{n+\lambda})[-] \right|$$

and hence

$$\left(T(A)_{n,\lambda}\right)^{C_{p^s}} \xleftarrow{\sim} \left| \left(sd_{p^s} THH^{(n)}(A; S^{n+\lambda})[-]\right)^{C_{p^s}} \right| .$$

Recall from last time that we

have a canonical homeomorphism

$$(\mathrm{sd}_r \, THH^{(n)}(A; S^{n+\lambda})[k])^{C_r} \xleftarrow{\sim}$$

$$\operatorname*{hocolim}_{(I^{(n)})^{k+1}} F((S^{j_0}\wedge\dots\wedge S^{j_k})^{\wedge r}, (\tilde{A}_{j_0}\wedge\dots\wedge\tilde{A}_{j_k})^{\wedge r}\wedge S^{n+\lambda})^{C_r}$$

where $C_r$ acts on $(\dots)^{\wedge r}$ by cyclically permuting the smash factors, on $S^{n+\lambda}$ by the action induced from the $C_r$-action on $\lambda$, and on the mapping space by conjugation. The map $r_p$ is then induced from the map

$$\rho_p^* F((S^{j_0}\wedge\dots\wedge S^{j_k})^{\wedge r}, (\tilde{A}_{j_0}\wedge\dots\wedge\tilde{A}_{j_k})^{\wedge r}\wedge S^{n+\lambda})^{C_p}$$

$$\to \rho_p^* F(((S^{j_0}\wedge\dots\wedge S^{j_k})^{\wedge r})^{C_p}, ((\tilde{A}_{j_0}\wedge\dots\wedge\tilde{A}_{j_k})^{\wedge r}\wedge S^{n+\lambda})^{C_p})$$

$$= F((S^{j_0}\wedge\dots\wedge S^{j_k})^{\wedge \bar{r}}, (\tilde{A}_{j_0}\wedge\dots\wedge\tilde{A}_{j_k})^{\wedge \bar{r}}\wedge S^{n+\rho_p^*\lambda^{C_p}})$$

induced from the inclusion

$$((S^{j_0}\wedge\dots\wedge S^{j_k})^{\wedge r})^{C_p} \hookrightarrow (S^{j_0}\wedge\dots\wedge S^{j_k})^{\wedge r}.$$

Here $\bar{r} = r/p$. Similar, the map $(r_p)^{C_{p^{s-1}}}$

is induced from the maps

$$F((S^{j_0}\wedge\cdots\wedge S^{j_k})^{\wedge n}, (\tilde{A}_{j_0}\wedge\cdots\wedge\tilde{A}_{j_k})^{\wedge n}\wedge S^{n+\lambda})^{G_{p^s}}$$

$$\to F((S^{j_0}\wedge\cdots\wedge S^{j_k})^{\wedge\bar{n}}, (\tilde{A}_{j_0}\wedge\cdots\wedge\tilde{A}_{j_k})^{\wedge\bar{n}}\wedge S^{n+\rho_p^*\lambda^{G_p}})^{G_{p^{s-1}}}$$

induced by the inclusion

$$(S^{j_0}\wedge\cdots\wedge S^{j_k})^{\wedge\bar{n}}$$

$$= ((S^{j_0}\wedge\cdots\wedge S^{j_k})^{\wedge n})^{G_p}$$

$$\subset (S^{j_0}\wedge\cdots\wedge S^{j_k})^{\wedge n}.$$

The fiber of this map is the equivariant mapping space

$$F\left((S^{j_0}\wedge\cdots\wedge S^{j_k})^{\wedge n}/(S^{j_0}\wedge\cdots\wedge S^{j_k})^{\wedge\bar{n}}, (\tilde{A}_{j_0}\wedge\cdots\wedge\tilde{A}_{j_k})^{\wedge n}\wedge S^{n+\lambda}\right)^{G_{p^s}}$$

Now for every (pointed) $G_{p^s}$-space $X$, the $G_p$-fixed set $X^{G_p}\subset X$ coincides with the singular set, i.e. the subset of points in $X$ with non-trivial isotropy subgroup. The reason is that every non-trivial subgroup of $C_{p^s}$ contains the subgroup $C_p \subset C_{p^s}$. Hence, for every (pointed) $C_{p^s}$-CW-complex $X$, the quotient $X/X^{C_p}$ is a $C_{p^s}$-CW-cx. where all cells are of orbit-type $C_{p^s}$, i.e. $X/X^{C_p}$ is a *free* pointed $C_{p^s}$-CW-complex. It follows (from the lemma we proved the time before last time) that the equivariant mapping space at hand is $m$-connected with

$$\begin{aligned} m &= n(\{1\}) - d(\{1\}) \\ &= jr + n + \dim(\lambda) - 1 - jr \\ &= n + \dim(\lambda) - 1. \end{aligned}$$

We shall use the map $r_p$ to understand the groups

$$\begin{aligned} TR^s_q(A;p) &= \pi_q^{C_{p^{s-1}}}(T(A)) \\ &= \operatorname*{colim}_{n,\lambda} \pi_q\left(F(S^{n+\lambda}, T(A)_{n,\lambda})^{C_{p^{s-1}}}\right). \end{aligned}$$

In general, if $EG$ is a free $G$-CW-complex that, non-equivariantly, is contractible, we let $\tilde{E}G$ be the $G$-CW-complex given as the mapping cone

$$EG_+ \xrightarrow{f} S^0 \to \tilde{E}G \to EG_+[-1]$$

where the map $f$ collapses $EG$ to the non-base-point of $S^0$. Then for closed subgroups $H \subset G$, we have

$$(EG_+)^H \simeq \begin{cases} \{+\} & H \neq \{1\} \\ S^0 & H = \{1\} \end{cases}$$

$$(\tilde{E}G)^H \simeq \begin{cases} S^0 & H \neq \{1\} \\ * & H = \{1\} \end{cases}.$$

If $T$ is a $G$-spectrum, we get an induced cofibration sequence of $G$-spectra

$$EG_+ \wedge T \to T \to \tilde{E}G \wedge T \to EG_+ \wedge T[-1]$$

which induces long-exact sequences

$$\cdots \to \pi_q^H(EG_+ \wedge T) \to \pi_q^H(T) \to \pi_q^H(\tilde{E}G \wedge T) \to \cdots$$

For $H$ finite, we write

$$\mathbb{H}_q(H, T) := \pi_q^H(EG_+ \wedge T).$$

There is a natural spectral sequence

$$E^2_{s,t} = H_s(H, \pi_t(T))$$

$$\Rightarrow \mathbb{H}_{s+t}(H, T).$$

It is obtained from the skeleton filtration of $EG$ (as an $H$-CW-cx.). We get an exact couple

$$E_{s,t} = \pi^H_{s+t}(gr_s EG \wedge T)$$

$$D_{s,t} = \pi^H_{s+t}(sk_s EG_+ \wedge T).$$

As an $H$-space, $gr_s EG$ is of the form $S \wedge H_+$, where $S$ is a bouquet of $s$-spheres. This implies that

$$\pi^H_{s+t}(gr_s EG \wedge T) \xrightarrow{\sim} \left(\pi_{s+t}(gr_s EG \wedge T)\right)^H$$

Moreover, we have isomorphisms

$$\pi_{s+t}(gr_s EG \wedge T) \xleftarrow{\sim} \pi_s(gr_s EG) \otimes \pi_t T$$

$$\xrightarrow{\sim} H_s(sk_s EG, sk_{s-1} EG) \otimes \pi_t T;$$

and the $H$-modules

$$P_s = H_s(sk_s EG, sk_{s-1} EG)$$

form a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}[H]$-module. Hence, as a cx.

$$E^1_{\cdot,t} = (P_\cdot \otimes \pi_t T)^H \underset{\sim}{\xleftarrow{N}} P_\cdot \underset{\mathbb{Z}[H]}{\otimes} \pi_t T$$

which shows that the $E^2$-term is as stated.

We return to the case of

$$TR^s_q(A;p) = \pi_q^{C_{p^{s-1}}}(T(A)).$$

<u>Prop</u> There is a canonical isomorphism

$$\pi_q^{C_{p^{s-1}}}(\tilde{E}\Pi \wedge T(A)) \xrightarrow{\sim} \pi_q^{C_{p^{s-2}}}(T(A)).$$

<u>Proof</u> The isomorphism is given by the

following composition.

$$\pi_q^{C_{p^{s-1}}}(\tilde{E}\wedge T(A)) = \operatorname*{colim}_{n,\lambda} \pi_q\left(F(S^{n+\lambda}, \tilde{E}\wedge T(A)_{n,\lambda})^{C_{p^{s-1}}}\right)$$

$$\xrightarrow{\sim} \operatorname*{colim}_{n,\lambda} \pi_q\left(\left(F(S^{n+\lambda^{C_p}}, (T(A)_{n,\lambda})^{C_p}\right)^{C_{p^{s-1}}/C_p}\right)$$

$$= \operatorname*{colim}_{n,\lambda} \pi_q\left(F(S^{n+\rho_p^*\lambda^{C_p}}, \rho_p^*(T(A)_{n,\lambda})^{C_p})^{C_{p^{s-2}}}\right)$$

$$\xrightarrow[\sim]{r_p} \operatorname*{colim}_{n,\lambda} \pi_q\left(F(S^{n+\rho_p^*\lambda^{C_p}}, T(A)_{n,\rho_p^*\lambda^{C_p}})^{C_{p^{s-2}}}\right)$$

$$\xrightarrow{\sim} \operatorname*{colim}_{n,\rho_p^*\lambda^{C_p}} \pi_q\left(F(S^{n+\rho_p^*\lambda^{C_p}}, T(A)_{n,\rho_p^*\lambda^{C_p}})^{C_{p^{s-2}}}\right)$$

$$\xrightarrow{\sim} \operatorname*{colim}_{n,\lambda} \pi_q\left(F(S^{n+\lambda}, T(A)_{n,\lambda})^{C_{p^{s-2}}}\right)$$

$$= \pi_q^{C_{p^{s-2}}}(T(A)).$$

The first map and the map $r_p$ are seen to be isom. by the lemma on the connectivity of an equiv. mapping space. The next to last map is defined as follows. Let

$$A_{n,\lambda} = \pi_q(F(S^{n+\lambda}, T(A)_{n,\lambda})^{C_{p^{s-2}}}).$$

The canonical maps

$$A_{n,\rho_p^*\lambda^{G_p}} \xrightarrow{\iota_{n,\rho_p^*\lambda^{G_p}}} \operatorname*{colim}_{n,\rho_p^*\lambda^{G_p}} A_{n,\rho_p^*\lambda^{G_p}}$$

are compatible:

$$\begin{array}{ccccc} \lambda & & \rho_p^*\lambda^{G_p} & & \\ \big\downarrow & \rightsquigarrow & \big\downarrow & \Rightarrow & \end{array}$$

$$\begin{tikzcd} \lambda \arrow[d] & \rho_p^*\lambda^{G_p} \arrow[d] & A_{n,\rho_p^*\lambda^{G_p}} \arrow[d] \arrow[dr, "\iota_{n,\rho_p^*\lambda^{G_p}}"] & \\ \lambda' & \rho_p^*\lambda'^{G_p} & A_{n,\rho_p^*\lambda'^{G_p}} \arrow[r, "\iota_{n,\rho_p^*\lambda'^{G_p}}"'] & \operatorname*{colim}_{n,\rho_p^*\lambda^{G_p}} A_{n,\rho_p^*\lambda^{G_p}} \end{tikzcd}$$

so define a map

$$\operatorname*{colim}_{n,\lambda} A_{n,\rho_p^*\lambda^{G_p}} \xrightarrow{f} \operatorname*{colim}_{n,\rho_p^*\lambda^{G_p}} A_{n,\rho_p^*\lambda^{G_p}}.$$

Similarly, the maps

$$A_{n,\rho_p^*\lambda^{G_p}} \xrightarrow{\iota_{n,\lambda^{G_p}}} \operatorname*{colim}_{n,\lambda} A_{n,\rho_p^*\lambda^{G_p}}$$

are compatible

$$\begin{tikzcd} \rho_p^*\lambda^{G_p} \arrow[d] & \lambda^{G_p} \arrow[d] & A_{n,\rho_p^*\lambda^{G_p}} \arrow[d] \arrow[dr, "\iota_{n,\lambda^{G_p}}"] & \\ \rho_p^*\lambda'^{G_p} & \lambda'^{G_p} & A_{n,\rho_p^*\lambda'^{G_p}} \arrow[r, "\iota_{n,\lambda'^{G_p}}"'] & \operatorname*{colim}_{n,\lambda} A_{n,\rho_p^*\lambda^{G_p}} \end{tikzcd}$$

so induces

$$\operatorname*{colim}_{n,\rho_p^*\lambda^{G_p}} A_{n,\rho_p^*\lambda^{G_p}} \xrightarrow{g} \operatorname*{colim}_{n,\lambda} A_{n,\rho_p^*\lambda^{G_p}}.$$

The composite $f \circ g$ is the identity, and the composite $g \circ f$ is equal to the self-map

$$\operatorname*{colim}_{n,\lambda} A_{n,\rho_p^* \lambda^{G_p}} \longrightarrow \operatorname*{colim}_{n,\lambda} A_{n,\rho_p^* \lambda^{G_p}}$$

given by the compatible system of maps

$$\begin{array}{ccccc} \lambda & & \lambda^{G_p} & & A_{n,\rho_p^*\lambda^{G_p}} \xrightarrow{c_{n,\lambda^{G_p}}} \\ \big\downarrow & \mapsto & \big\downarrow & \Rightarrow & \big\downarrow \qquad\qquad\qquad \operatorname*{colim}_{n,\lambda} A_{n,\rho_p^*\lambda^{G_p}} . \\ \lambda' & & \lambda'^{G_p} & & A_{n,\rho_p^*\lambda'^{G_p}} \xrightarrow[c_{n,\lambda'^{G_p}}]{} \end{array}$$

But this map is equal to the identity map since

$$\begin{array}{ccc} \lambda^{G_p} & & A_{n,\rho_p^*\lambda^{G_p}} \xrightarrow{c_{n,\lambda^{G_p}}} \\ \big\downarrow & \Rightarrow & \Vert \qquad\qquad\qquad \operatorname*{colim}_{n,\lambda} A_{n,\rho_p^*\lambda^{G_p}} . \\ \lambda & & A_{n,\rho_p^*\lambda^{G_p}} \xrightarrow[c_{n,\lambda}]{} \end{array}$$

The last map is actually an identity since every $\lambda$ is of the form $\rho_p^* \lambda'^{G_p}$ with $\lambda' = (\rho_p^{-1})^* \lambda$. //

We obtain the fundamental sequence

$$\cdots \to \mathbb{H}_q(G_{p^{n-1}}, T(A)) \xrightarrow{N} TR^n_q(A;p) \xrightarrow{R} TR^{n-1}_q(A;p) \to \cdots$$

Let $A$ be a commutative ring, and let $p$ be a prime. The ring $W_n(A)$ of ($p$-typical) Witt vectors of length $n$ in $A$ is the set of $n$-tuples $a = (a_0, a_1, \dots, a_{n-1})$ in $A$, but with a new ring structure characterized by the requirement that the ghost map

$$W_n(A) \xrightarrow{w} A^n$$

$$(a_0, a_1, \dots, a_{n-1}) \longmapsto (w_0, w_1, \dots, w_{n-1})$$

$$w_s = a_0^{p^s} + p a_1^{p^{s-1}} + \dots + p^s a_s$$

be a natural transformation of functors from comm. rings to comm. rings. Here the ring-operations on $A^n$ are component-wise. We shall now construct the ring-operations on $W_n(A)$. First note that the ghost map is 1-1 if $p$ is a non-zero-divisor in $A$.

<u>Lemma</u> (Dwork) Suppose that there exists a ring-homomorphism $f: A \to A$ s.t. $f(a) \equiv a^p \pmod{pA}$, for all $a \in A$. Then a sequence $(x_0, x_1, \dots, x_{n-1}) \in A^n$ is in the image of the ghost map if and only if for all $0 < s < n$,

$$x_s \equiv f(x_{s-1}) \pmod{p^s A}.$$

<u>Proof</u> It follows from the binomial formula that if $x \equiv y \pmod{pA}$, then $x^{p^s} \equiv y^{p^s} \pmod{p^{s+1}A}$. To prove the "only if" statement, we calculate

$$f(w_{s-1}) = f(a_0^{p^{s-1}} + p a_1^{p^{s-2}} + \dots + p^{s-1} a_{s-1})$$

$$= f(a_0)^{p^{s-1}} + p f(a_1)^{p^{s-2}} + \dots + p^{s-1} f(a_{s-1})$$

$$\equiv a_0^{p^s} + p a_1^{p^{s-1}} + \dots + p^{s-1} a_{s-1}^p \pmod{p^s A}$$

$$\equiv w_s \pmod{p^s A}.$$

Conversely, given $(x_0, x_1, \dots, x_{n-1})$ s.t.

$x_s \equiv f(x_{s-1}) \pmod{p^s A}$, we can choose $(a_0, a_1, \ldots, a_{n-1})$ inductively s.t.

$$p^s a_s = x_s - f(x_{s-1}).$$

//

We now construct the addition on $W_n(A)$. Let $A = \mathbb{Z}[a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}]$, and let $f : A \to A$ be the ring-homomorphism given by $f(a_s) = a_s^p$ and $f(b_s) = b_s^p$, $0 \le s < n$. Then $f(x) \equiv x^p \pmod{pA}$, for all $x \in A$. Let

$$a = (a_0, a_1, \ldots, a_{n-1}) \in W_n(A)$$

$$b = (b_0, b_1, \ldots, b_{n-1}) \in W_n(A).$$

We use the lemma of Dwork to show that there exists

$$s = (s_0, s_1, \ldots, s_{n-1}) \in W_n(A)$$

s.t.

$$w(s) = w(a) + w(b).$$

We must check that

$$w_s(a) + w_s(b) \equiv f(w_{s-1}(a) + w_{s-1}(b)) \pmod{p^s A}$$

and this is straightforward. The vector $s$ is unique, since $p$ is a non-zero-divisor in $A$. Now, if $A'$ is any comm. ring, and if

$$a' = (a_0', a_1', \ldots, a_{n-1}') \in W_n(A')$$

$$b' = (b_0', b_1', \ldots, b_{n-1}') \in W_n(A'),$$

then there exist a unique ring-homomorphism $g : A \to A'$ s.t. $g(a_s) = a_s'$ and $g(b_s) = b_s'$, $0 \le s < n$. We then define $a' + b'$ to be the vector

$$s' = (s_0', s_1', \ldots, s_{n-1}') \in W_n(A')$$

with $s_i' = g(s_i)$, $0 \le i < n$.

To show that the addition defined above is associative, we consider

$$A = \mathbb{Z}[a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}, c_0, \ldots, c_{n-1}]$$

and

$$a = (a_0, a_1, \ldots, a_{n-1}) \in W_n(A)$$

$$b = (b_0, b_1, \ldots, b_{n-1}) \in W_n(A)$$

$$c = (c_0, c_1, \ldots, c_{n-1}) \in W_n(A).$$

By naturality, it suffices to show that in $W_n(A)$,

$$(a+b)+c = a+(b+c).$$

If we apply the ghost map to this equation, we get, because of the way the addition was defined,

$$(w(a)+w(b))+w(c) = w(a)+(w(b)+w(c)).$$

This is true, since the addition in $A^n$ is associative. Since $p$ is a non-zero-divisor in $A$, the ghost map $w: W_n(A) \to A^n$ is injective. Hence, the equation $(a+b)+c = a+(b+c)$ holds in $W_n(A)$. So addition is associative.

The remaining ring-operations are constructed similarly, and the ring-axioms are checked in the same way, too. Let us calculate a few of the sum polynomials

$$s = (s_0, s_1, \ldots, s_{n-1}).$$

By definition, we have

$$s_0 = a_0 + b_0$$

$$s_0^p + p s_1 = a_0^p + p a_1^p + b_0^p + p b_1$$

$$s_0^{p^2} + p s_1^p + p^2 s_2 = a_0^{p^2} + p a_1^p + p^2 a_2 + b_0^{p^2} + p b_1^p + p^2 b_2$$

$$\vdots$$

so

$$s_0 = a_0 + b_0$$

$$s_1 = a_1 + b_1 + \frac{a_0^p + b_0^p - (a_0 + b_0)^p}{p}$$

$$s_2 = a_2 + b_2 +$$

$$\frac{a_0^{p^2} + b_0^{p^2} - (a_0 + b_0)^{p^2} + p\left(a_1^p + b_1^p - \left(a_1 + b_1 + \frac{a_0^p + b_0^p - (a_0 + b_0)^p}{p}\right)^p\right)}{p^2}$$

$$s_3 = \cdots$$

The lemma of Dwork encodes all the congruences necessary to define the $s_i$.

The projection on the first $n-1$ factors defines a ring-homomorphism

$$W_n(A) \xrightarrow{R} W_{n-1}(A)$$

called restriction. There is a second ring-homomorphism

$$W_n(A) \xrightarrow{F} W_{n-1}(A)$$

called Frobenius that is characterized by the requirement that

$$w(F(a_0, \dots, a_{n-1})) = (w_1(a), \dots, w_{n-1}(a)),$$

and a map of $W_n(A)$-modules

$$F_* W_{n-1}(A) \xrightarrow{V} W_n(A)$$

called Verschiebung given by

$$V(a_0, \dots, a_{n-2}) = (0, a_0, \dots, a_{n-2}).$$

Finally, there is a multiplicative map

$$A \xrightarrow{[-]_n} W_n(A)$$

called the Teichmüller representative

given by

$$[a]_n = (a, 0, \dots, 0).$$

In particular, $[1]_n = (1, 0, \dots, 0)$ is the multiplicative unit in $W_n(A)$. The following relations hold

$$F([a]_n) = [a]_{n-1}^p$$

$$FV(a) = p \cdot a,$$

where $p = [1]_n + \dots + [1]_n$ ($p$ times). The following formula is useful.

$$(a_0, a_1, \dots, a_{n-1}) = \sum_{0 \le s < n} V^s([a_s]_{n-s}).$$

The relations above are all proved by considering the universal case and verifying that the relation holds after applying the ghost map.

Last time we proved that for $q=0$, the homotopy group

$$TR^n_q(A;p) = [S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+, T(A)]_{\mathbb{T}}$$

is canonically isomorphic to the ring of Witt vectors $W_n(A)$. We shall now examine the structure of the higher homotopy groups. Suppose that $A$ is a $\mathbb{Z}_{(p)}$-algebra, where $p$ is odd. Then these groups form a <u>Witt</u> <u>complex</u> over $A$. This, by definition, is a quadruple

$$(E, \lambda, F, V)$$

where

(i) $E = \{E^{\cdot}_n\}_{n \in \mathbb{N}}$ is a pro-differential graded ring,

(ii) $\lambda$ is a strict map of pro-rings

$$W_n(A) \xrightarrow{\lambda} E^0_n,$$

(iii) $F$ is a strict map of pro-graded rings

$$E_n^{\circ} \xrightarrow{F} E_{n-1}^{\circ},$$

(iv) $V$ is a strict map of pro-graded abelian groups

$$E_{n-1}^{\circ} \xrightarrow{V} E_n^{\circ},$$

such that the following relations hold.

$$F\lambda = \lambda F$$

$$F d\lambda[a]_n = \lambda[a]_{n-1}^{p-1} d\lambda[a]_{n-1}$$

$$V\lambda = \lambda V$$

$$x V(y) = V(F(x) y)$$

$$FV = p$$

$$F dV = d.$$

A map of Witt complex is a

strict map of pro-differential graded rings $f : E_n^{\cdot} \to E_n'^{\cdot}$ s.t. $\lambda' = f\lambda$, $F'f = fF'$ and $V'f = fV$.

<u>Lemma</u> The following relations hold.

$$dF = pFd$$

$$Vd = pdV$$

<u>Proof</u> let $x, y \in E_n^{\cdot}$ and note that

$$V(x\,dy) = V(xFdV(y)) = V(x)\,dV(y)$$

Hence,

$$dF(x) = FdVF(x) = Fd(V(1).x)$$

$$= F(dV(1).x + V(1).dx)$$

$$= FdV(1).F(x) + FV(1).Fd(x)$$

$$= d(1).F(x) + pFd(x)$$

$$= pFd(x)$$

and

$$V d(x) = V(1) . dV(x)$$

$$= d(V(1).V(x)) - dV(1).V(x)$$

$$= dV(FV(1).x) - V(FdV(1).x)$$

$$= dV(px) - V(d(1).x)$$

$$= p\, dV(x).$$

//

It turns out that there are (infinitely) many Witt complexes over A. However, as we shall soon see, the category of Witt complexes over A has an initial object. We first consider an easier situation.

Consider the category of differential graded rings $E^\cdot$ together with a ring-homomorphism

$$A \xrightarrow{\lambda} E^0.$$

The initial object of this category is called the de Rham complex of A and written $\Omega^{\cdot}_A$. It is constr. as follows. Let

$$A \xrightarrow{d} \Omega^1_A$$

be the initial example of a derivation of A into an A-module. The A-module $\Omega^1_A$ is generated by symbols $da$, $a \in A$, subject to the relations that for all $a, b \in A$,

$$d(a+b) = da + db$$

$$d(ab) = b\,da + a\,db .$$

Then, as a graded A-algebra,

$$\Omega^{\cdot}_A = \Lambda^{\cdot}_A(\Omega^1_A)$$

and the differential is given by

$$\Omega^q_A \xrightarrow{d} \Omega^{q+1}_A$$

$$a_0\,da_1 \dots da_q \longmapsto da_0\,da_1 \dots da_q .$$

The initial Witt complex over $A$ is not constructed in such an explicit fashion. Instead we use.

<u>Lemma</u> (Freyd) Let $\mathcal{C}$ be a category which has all small limits and which satisfies the <u>solution set condition</u>: There is a <u>set</u> of objects $k_i$, $i \in I$, in $\mathcal{C}$ s.t. for every object $c$ in $\mathcal{C}$ there exists a morphism $k_i \to c$ in $\mathcal{C}$, for some $i \in I$. Then $\mathcal{C}$ has an initial obj.

<u>Pf</u> (Sketch) Since $I$ is a set and since $\mathcal{C}$ has all small limits, we can form the product

$$k = \prod_{i \in I} k_i .$$

If $c$ is an obj. in $\mathcal{C}$, then, by the solution set condition, there exists a morphism $k \to c$, but this need not be unique. In particular, the set $\mathrm{End}_{\mathcal{C}}(k)$ of endomorphisms of $k$ may contain more than the identity morphism. Let $d$ be the equalizer

$$d \longrightarrow k \underset{f}{\overset{\Delta}{\rightrightarrows}} \prod_{\varphi \in \mathrm{End}_{\mathcal{E}}(k)} k$$

where $(f)_\varphi = \varphi$. It exists since $\mathcal{E}$ has all small limits. Then $d$ is an initial object of $\mathcal{E}$. //

<u>Thm</u> The category of Witt complexes over $A$ has an initial object called the de Rham-Witt complex and written $W_n\Omega^{\cdot}_A$. In addition, the canonical map

$$\Omega^{\cdot}_{W_n(A)} \xrightarrow{\sim} W_n\Omega^{\cdot}_A$$

is surjective.

Pf The category of Witt complexes over A has all small limits, so we must show that the solution set cond. is satisfied. We will show that if $(E, \lambda, F, V)$ is a Witt complex over A, then the image of the canonical map

$$\Omega^{\bullet}_{W_n(A)} \xrightarrow{\lambda} E^{\bullet}_n$$

forms a sub-Witt complex of E. The image is canonically isom. to the quotient of $\Omega^{\bullet}_{W_n(A)}$ by the kernel of $\lambda$. Since there is (only) a set of such quotients, the solution set cond. is satisfied.

It remains to show that the image of $\lambda$ is a sub-Witt complex of E, or equivalently, that the image is preserved by

$F$ and $V$. This is easy for $V$:

$$V(\lambda(x_0)\, d\lambda(x_1) \cdots d\lambda(x_q))$$
$$= V(\lambda(x_0))\, dV(\lambda(x_1)) \cdots dV(\lambda(x_q))$$
$$= \lambda(V(x_0))\, d\lambda(V(x_1)) \cdots d\lambda(V(x_q)).$$

Since $F$ is multiplicative, it is enough to show that for all $a \in W_n(A)$, $F\,d\lambda(a)$ is in the image. Now

$$a = [a_0]_n + V([a_1]_{n-1}) + \cdots + V^{n-1}([a_{n-1}]_1)$$

so

$$F d\lambda(a) = F d\lambda([a_0]_n) + F dV\lambda([a_1]_{n-1}) + \cdots$$
$$\cdots + F dV^{n-1}\lambda([a_{n-1}]_1)$$
$$= \lambda([a_0]_{n-1}^{p-1})\, d\lambda[a_0]_{n-1} + d\lambda([a_1]_{n-1}) + \cdots$$
$$\cdots + dV^{n-2}\lambda([a_{n-1}]_1)$$

which is in the image.

Finally, we show that the map

$$\Omega^\cdot_{W_n(A)} \xrightarrow{\lambda} W_n\Omega^\cdot_A$$

is surjective. The image $E^\cdot_n$ of this map forms a sub-Witt complex of $W_n\Omega^\cdot_A$. Hence, there is a unique map of Witt complexes

$$W_n\Omega^\cdot_A \xrightarrow{f} E^\cdot_n .$$

The composite

$$W_n\Omega^\cdot_A \xrightarrow{f} E^\cdot_n \hookrightarrow W_n\Omega^\cdot_A$$

is an endom. of the initial obj. $W_n\Omega^\cdot_A$ and hence is the identity map. Therefore, the inclusion

$$E^\cdot_n \hookrightarrow W_n\Omega^\cdot_A$$

is surjective. //

4/29/04

We evaluate the de Rham-Witt cx. of the polynomial ring

$$A = \mathbb{F}_p[T_1, \dots, T_n].$$

Let

$$B = \mathbb{Z}_p[T_1, \dots, T_n]$$

$$C = \bigcup_{r \geq 0} \mathbb{Q}_p[T_1^{p^{-r}}, \dots, T_n^{p^{-r}}],$$

and let $\mathbb{N}[\frac{1}{p}]$ (resp. $\mathbb{N}_0[\frac{1}{p}]$) be the set of all positive (resp. non-neg.) rational numbers whose denominator is a power of p. We consider the diff. graded algebra

$$\Omega^{\cdot}_{C/\mathbb{Q}_p} \xleftarrow{\sim} \bigcup_{r \geq 0} \Omega^{\cdot}_{\mathbb{Q}_p[T_1^{p^{-r}}, \dots, T_n^{p^{-r}}]/\mathbb{Q}_p}.$$

We have

$$\Omega^{\cdot}_{\mathbb{Q}_p[T_1^{p^{-r}}, \dots, T_n^{p^{-r}}]/\mathbb{Q}_p}$$

$$= \Lambda^{\cdot}_{\mathbb{Q}_p[T_1^{p^{-r}}, \dots, T_n^{p^{-r}}]}\{d(T_1^{p^{-r}}), \dots, d(T_n^{p^{-r}})\}.$$

We note that formally

$$d(T_i^{p^{-r}}) = p^{-r} T_i^{p^{-r}} d\log T_i .$$

So we can write every elem.

$$\omega \in \Omega^m_{\mathbb{Q}_p[T_1^{p^{-r}}, \dots, T_n^{p^{-r}}] / \mathbb{Q}_p}$$

uniquely as a sum

$$\omega = \sum_{1 \le i_1 < \dots < i_m \le n} a_{i_1, \dots, i_m} \cdot d\log T_{i_1} \cdots d\log T_{i_m}$$

where $a_{i_1, \dots, i_m} \in \mathbb{Q}_p[T_1^{p^{-r}}, \dots, T_n^{p^{-r}}]$ is divisible by $T_{i_s}^{p^{-r}}$, for all $1 \le s \le m$. The advantage of this description is that it is easy to evaluate the map induced by the inclusion

$$\mathbb{Q}_p[T_1^{p^{-r}}, \dots, T_n^{p^{-r}}] \hookrightarrow \mathbb{Q}_p[T_1^{p^{-(r+1)}}, \dots, T_n^{p^{-(r+1)}}] .$$

Indeed, it is given simply by the inclusion on the coeff. polynomials $a_{i_1, \dots, i_m}$. Hence, every elem.

$$\omega \in \Omega^m_{C/\mathbb{Q}_p}$$

can be written uniquely

$$\omega = \sum_{1 \le i_1 < \dots < i_m \le n} a_{i_1,\dots,i_m} \cdot d\log T_{i_1} \dots d\log T_{i_m} ,$$

where $a_{i_1,\dots,i_m} \in C$ is divisible by $T_{i_s}^{p^{-r}}$, for all $1 \le s \le m$ and for some $r \ge 0$. We call the polynomials $a_{i_1,\dots,i_m} \in C$, $1 \le i_1 < \dots < i_m \le n$, the <u>coordinates</u> of $\omega$. We say that a polynomial $a \in C$ is <u>integral</u>, if it has coeff. in $\mathbb{Z}_p \subset \mathbb{Q}_p$, and we say that $\omega \in \Omega^m_{C/\mathbb{Q}_p}$ is <u>integral</u>, if all the coordinates of $\omega$ are integral. We define

$$E^m \subset \Omega^m_{C/\mathbb{Q}_p}$$

to be the subset of those $\omega$ s.t. both $\omega$ and $d\omega$ are integral. Then

$$E^\cdot \subset \Omega^\cdot_{C/\mathbb{Q}_p}$$

is a sub-differential graded algebra.

The $\mathbb{Q}_p$-algebra automorphism

$$C \xrightarrow[\sim]{\Phi} C$$

$$T_i^{p^{-r}} \longmapsto T_i^{p^{-(r-1)}}$$

induces an isom. of differential graded algebras

$$\Omega^{\cdot}_{C/\mathbb{Q}_p} \xrightarrow{\Phi^{\cdot}} \Omega^{\cdot}_{C/\mathbb{Q}_p}.$$

The maps

$$\Omega^{m}_{C/\mathbb{Q}_p} \xrightarrow{F = p^{-m}\Phi^{m}} \Omega^{m}_{C/\mathbb{Q}_p}$$

form autom. of graded algebras

$$\Omega^{\cdot}_{C/\mathbb{Q}_p} \xrightarrow{F} \Omega^{\cdot}_{C/\mathbb{Q}_p},$$

and this restricts to a map of graded algebras

$$E^{\cdot} \xrightarrow{F} E^{\cdot}.$$

The automorphism of graded $C$-mod.

$$\Omega^{\cdot}_{C/\mathbb{Q}_p} \xrightarrow{V = pF^{-1}} \Omega^{\cdot}_{C/\mathbb{Q}_p}$$

also restricts to a map of graded $\mathbb{Z}_p$-modules

$$E^{\cdot} \xrightarrow{V} E^{\cdot}.$$

Indeed, if

$$\omega = \sum_{i_1 < \cdots < i_m} a_{i_1, \ldots, i_m} \, d\log T_{i_1} \cdots d\log T_{i_m},$$

then

$$F\omega = \sum_{i_1 < \cdots < i_m} \Phi(a_{i_1, \ldots, i_m}) \, d\log T_{i_1} \cdots d\log T_{i_m}$$

$$V\omega = \sum_{i_1 < \cdots < i_m} p\,\Phi^{-1}(a_{i_1, \ldots, i_m}) \, d\log T_{i_1} \cdots d\log T_{i_m}.$$

By definition,

$$dF = pFd$$

$$Vd = pdV,$$

and

$$\omega V(\omega') = V(F(\omega)\omega').$$

Prop (a) The element

$$x = \sum_{k \in \mathbb{N}_0[\frac{1}{p}]^n} a_k T_1^{k_1} \cdots T_n^{k_n} \in C$$

belongs to $E^0 \subset C$ if and only if for all $k$, $a_k \in \mathbb{Z}_p$ and the denominator of $k_i$ divides $a_k$, $1 \le i \le n$.

(b) We have:

(i) $$E^0 = \sum_{r \geq 0} V^r B$$

(ii) $$\bigcap_{r \geq 0} V^r E^0 = 0$$

(iii) $$B \cap V^r E^0 = p^r B.$$

(c) The $\mathbb{Z}_p$-algebra homomorphism

$$B \longrightarrow W(A)$$

$$T_i \longmapsto [T_i]$$

extends uniquely to a $\mathbb{Z}_p$-algebra homomorphism

$$E^0 \xrightarrow{\tau} W(A)$$

s.t. $V\tau = \tau V$. Moreover, $\tau$ is injective and induces isomorphisms

$$E^0 / V^r E^0 \xrightarrow{\sim} W_r(A).$$

Proof (a) This is clear.

(b) If $x = a T_1^{k_1} \cdots T_n^{k_n} \in E^0$, and if

$$p^s = \max\{ \mathrm{den}(k_i) \mid 1 \le i \le n \},$$

then $x = V^s(y)$, where $y = p^{-s} a T_1^{p^s k_1} \cdots T_n^{p^s k_n}$, so (i) follows. If

$$x = \sum a_k T_1^{k_1} \cdots T_n^{k_n} \in V^r E^0$$

then $p^r$ divides $a_k$, for all $k \in \mathbb{N}_0[\frac{1}{p}]^n$. This proves (ii) and (iii).

(c) The uniqueness of $c$ follows from part (b) (i). To prove existence, we let

$$\bar{A} = \bigcup_{r \geq 0} \mathbb{F}_p[T_1^{p^{-r}}, \ldots, T_n^{p^{-r}}]$$

$$\bar{B} = \bigcup_{r \geq 0} \mathbb{Z}_p[T_1^{p^{-r}}, \ldots, T_n^{p^{-r}}].$$

Then $\bar{A}$ is a perfect $\mathbb{F}_p$-algebra, the $\mathbb{Z}_p$-algebra automorphism

$$\begin{array}{ccc} \bar{B} & \xrightarrow{f} & \bar{B} \\ T_i^{p^{-r}} & \longmapsto & T_i^{p^{-(r-1)}} \end{array}$$

lifts the Frob. of $\bar{A} = \bar{B}/p\bar{B}$, and $p$ is a non-zero-divisor in $\bar{B}$. Hence, we have the ring-homom.

$$\bar{B} \xrightarrow{t_f} W(\bar{A})$$

which satisfies

$$t_f(T_i^{p^{-r}}) = [T_i^{p^{-r}}]$$

$$F(t_f(x)) = t_f(f(x)).$$

The Frob. $F : W(\bar{A}) \to W(\bar{A})$ is equal to the map induced by the Frob. of $\bar{A}$, and hence is an isom. so

$$W(\bar{A}) \xrightarrow{V = pF^{-1}} W(\bar{A}) .$$

Let $V = pf^{-1} : \bar{B} \to \bar{B}$. Then we have $V t_f(x) = t_f(Vx)$. Finally, $E^0 \subset \bar{B}$ and $W(A) \subset W(\bar{A})$ compatible with $V$ (and $F$). It follows that

$$\begin{array}{ccc} \bar{B} & \xrightarrow{t_f} & W(\bar{A}) \\ \uparrow & & \uparrow \\ E^0 & \xrightarrow{\tau} & W(A) \end{array} .$$

By (b) (iii), the $V^r : B \to E^0$ induces a map of $A$-algebras

$$\varphi^r_* A \xrightarrow{V^r} gr^r_V E^0$$

and the following diagr. commutes.

$$\begin{tikzcd} & \varphi^r_* A \arrow[dl, "V^r"'] \arrow[dr, "V^r", "\sim"'] & \\ \mathrm{gr}^r_V E^\circ \arrow[rr, "\mathrm{gr}^r_V z"] & & \mathrm{gr}^r_V W(A) \end{tikzcd}$$

We wish to show that the lower map is an iso., and since the right-hand map $V^r$ is an isom., it is enough to show that the left-hand map $V^r$ is an isom. Since the map $V: E^\circ \to E^\circ$ is injective, it suffices to prove this for $r = 0$. This is the statement that

$$B/pB \xrightarrow{\sim} E^\circ / VE^\circ$$

which follows from (b) (i) and (iii). //

We consider the filtr.

$$E^\bullet = \mathrm{Fil}^0 E^\bullet \supset \mathrm{Fil}^1 E^\bullet \supset \cdots$$

of $E^{\cdot}$ by the differential graded ideals

$$\mathrm{Fil}^n E^q = V^n E^q + dV^n E^{q-1}$$

and define

$$E^{\cdot}_n = E^{\cdot} / \mathrm{Fil}^n E^{\cdot} .$$

The operators $F$ and $V$ induce maps of pro-graded rings and pro-graded abelian groups, resp.,

$$E^{\cdot}_n \xrightarrow{F} E^{\cdot}_{n-1}$$

$$E^{\cdot}_{n-1} \xrightarrow{V} E^{\cdot}_n$$

which satisfy $FV = p$, $dF = pFd$, and $Vd = pdV$. The inverse of the isom. $\tau$ of the prop., gives an isom. of pro-rings

$$W_n(A) \xrightarrow[\sim]{\lambda} E^0_n .$$

To show that $(E, \lambda, F, V)$ is a Witt complex it remains to verify the following two relations.

$$F\, d\, \gamma[a]_n = \gamma[a]_{n-1}^{p-1}\, d\, \gamma[a]_{n-1} \qquad a \in A$$

$$F d V = d\,.$$

We recall that, after mult. by $p$, the two relations are a consequence of the other relations. Hence, the two rel. follow if we prove that the limit

$$\hat{E}^{\cdot} = \lim_{n} E_n^{\cdot}$$

is $p$-torsion free. We shall do so after an analysis of the structure of $E_n^{\cdot}$ and the various maps. The more careful analysis will show.

<u>Thm</u> (Deligne) The canonical map

$$W_n \Omega_A^{\cdot} \xrightarrow{\gamma} E_n^{\cdot}$$

is an isomorphism.