Homotopy Theory

Definition. A *model category* is a category $\mathcal{C}$ together with three classes of maps called the *weak equivalences*, the *fibrations*, and the *cofibrations*, each of which is closed under composition and contains all identity maps, and such that the following axioms hold.

(M1) All small limits and colimits exists in $\mathcal{C}$.

(M2) If f and g are composable maps in $\mathcal{C}$, and if two out of three of the maps f, g, and gf are weak equivalences, then so is the third.

(M3) If f is a retract of g, and if g is a fibration, cofibration, or weak equivalence, then so is f.

(M4) Given a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y, \end{array}$$

then there exists a map $h\colon B \to X$ such that $ph = g$ and $hi = f$, if one of the following two conditions is satisfied.

(i) i is a cofibration, and p is both a fibration and a weak equivalence.

(ii) i is both a cofibration and a weak equivalence, and p is a fibration.

(M5) Every map f can be factored in the following two ways.

(i) $f = pi$, where i is a cofibration, and where p is both a fibration and a weak equivalence.

(ii) $f = pi$, where i is both a cofibration and a weak equivalence, and where p is a fibration.

A map that is both a fibration and a weak equivalence is called a *trivial fibration*, and a map that is both a cofibration and a weak equivalence is called a *trivial cofibration*.

Definition. Let $\mathcal{C}$ be a category such that all small colimits exist, and let

$$I = \{U \to V\}$$

be a class of maps in $\mathcal{C}$.

(i) An object X of $\mathcal{C}$ is *small relative to I* if for every sequence

$$Y_1 \xrightarrow{f_1} Y_2 \xrightarrow{f_2} Y_3 \xrightarrow{f_3} \dots$$

of maps in I, the canonical map

$$\operatorname*{colim}_i \operatorname{Hom}_{\mathcal{C}}(X, Y_i) \to \operatorname{Hom}_{\mathcal{C}}(X, \operatorname*{colim}_i Y_i)$$

is a bijection.

(ii) A map p is an *I-injective* if it has the right lifting property with respect to all maps in I.

(iii) A map i is an *I-cofibration* if it has the left lifting property with respect to all I-injective maps.

(iv) A map $f\colon A \to B$ is an *I-cellular* map if there exists a sequence of maps

$$A = B_0 \xrightarrow{i_0} B_1 \xrightarrow{i_1} B_2 \to \dots$$

together with push-out squares

$$\begin{array}{ccc} \sqcup_\alpha U_\alpha & \longrightarrow & B_{m-1} \\ \downarrow & & \downarrow{\scriptstyle i_{m-1}} \\ \sqcup_\alpha V_\alpha & \longrightarrow & B_m \end{array}$$

and maps $f_m\colon B_m \to B$ such that $f_0 = f$ and $f_m i_{m-1} = f_{m-1}$ and such that the induced map

$$\operatorname*{colim}_m B_m \to B$$

is an isomorphism.

PROPOSITION (Small object argument). *Let $\mathcal{C}$ be a category in which all small colimits exist. Let I be a* set *of maps in $\mathcal{C}$ and assume that the domains of the maps in I are small relative to the class of I-cellular maps. Then every map f in $\mathcal{C}$ can be factored as $f = pi$, where i is an I-cellular map, and where p is an I-injective.*

LEMMA. *Every I-cellular map is an I-cofibration and every I-cofibration is a retract of an I-cellular map.*

DEFINITION. A model category $\mathcal{C}$ is *cofibrantly generated* if there exists sets of maps I and J such that the following holds.

(i) The domains of the maps in I are small relative to the I-cellular maps.

(ii) The domains of the maps in J are small relative to the J-cellular maps.

(iii) The fibrations are the J-injective maps.

(iv) The trivial fibrations are the I-injective maps.

We say that I is a set of *generating cofibrations* and that J is a set of *generating trivial cofibrations*.

PROPOSITION. *Let $\mathcal{C}$ be a cofibrantly generated model category, let I be a set of generating cofibrations, and let J be a set of generating trivial cofibrations. Then the cofibrations are the I-cofibrations and the trivial cofibrations are the J-cofibrations.*

THEOREM. *Let $\mathcal{C}$ be a category in which all small limits and colimits exist, and let $\mathcal{W}$ be a class of maps in $\mathcal{C}$ that is closed under retracts and satisfies the "two out of three" axiom (M2). Let I and J be two sets of maps in $\mathcal{C}$ and assume that the following (i)–(iv) holds.*

(i) The domains of the maps in I are small relative to the class of I-cellular maps. The domains of J are small relative to the class of J-cellular maps.

(ii) Every J-cofibration is both an I-cofibration and an element of $\mathcal{W}$.

(iii) Every I-injective is both a J-injective and an element of $\mathcal{W}$.

(iv) One of the following two conditions (a)–(b) holds:

(a) A map that is both an I-cofibration and an element of $\mathcal{W}$ is a J-cofibration.

(b) A map that is both a J-injective and an element of $\mathcal{W}$ is an I-injective.

Then $\mathcal{C}$ has a cofibrantly generated model structure, where $\mathcal{W}$ is the class of weak equivalences, where I is a set of generating cofibrations, and where J is a set of generating fibrations.

Prove the following result of D. Kan.

THEOREM. *Let $\mathcal{C}$ be a cofibrantly generated model category, let I be a set of generating cofibrations, and let J be a set of generating trivial cofibrations. Let $\mathcal{D}$ be a category in which all small limits and colimits exists, and let (F, G, φ) be an adjunction between $\mathcal{C}$ and $\mathcal{D}$. Let FI and FJ be the sets of maps in $\mathcal{D}$ defined by*

$$FI = \{F(u) \mid u \in I\}$$
$$FJ = \{F(v) \mid v \in J\}$$

and suppose that the following (i)–(ii) holds.

(i) The domains of the maps in FI are small relative to the class of FI-cellular maps. The domains of the maps in FJ are small relative to the class of FJ-cellular maps.

(ii) The functor G takes FJ-cellular maps to weak equivalences in $\mathcal{C}$.

Then there exists a cofibrantly generated model structure on the category $\mathcal{D}$, where a map f is a weak equivalence if and only if $G(f)$ is a weak equivalence in $\mathcal{C}$, where FI is a set of generating cofibrations, and where FJ is a set of generating trivial cofibrations. Moreover, the adjunction (F, G, φ) is a Quillen adjunction.

This theorem is very useful for producing model structures. Here is an example.

EXAMPLE. Let R be a ring, let $\mathcal{C}$ be the category of simplicial sets, and let $\mathcal{D}$ be the category of simplicial left R-modules. Let (F, G, φ) be the adjunction between $\mathcal{C}$ and $\mathcal{D}$, where F is the functor that to a simplicial set X associates the simplicial left R-module $F(X)$, where $F(X)_n$ is the free left R-module generated by the set X_n, and where G is the forgetful functor that to a simplicial left R-module associates the underlying simplicial set. Then the two conditions (i) and (ii) of the theorem above are satisfied such that we obtain a cofibrantly generated model structure on the category of simplicial left R-modules.