The left and right homotopy relations

We recall that a *coproduct* of two objects A and B in a category $\mathcal{C}$ is an object $A \amalg B$ together with two maps $\mathrm{in}_1 \colon A \to A \amalg B$ and $\mathrm{in}_2 \colon B \to A \amalg B$ such that, for every pair of maps $f \colon A \to C$ and $g \colon B \to C$, there exists a *unique* map

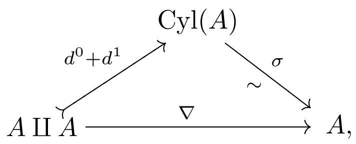
$$f + g \colon A \amalg B \to C$$

such that $f = (f+g) \circ \mathrm{in}_1$ and $g = (f+g) \circ \mathrm{in}_2$. If both $A \amalg B$ and $A \amalg' B$ are coproducts of A and B, then the maps $\mathrm{in}_1' + \mathrm{in}_2' \colon A \amalg B \to A \amalg' B$ and $\mathrm{in}_1 + \mathrm{in}_2 \colon A \amalg' B \to A \amalg B$ are isomorphisms and each others inverses. The map $\nabla = \mathrm{id} + \mathrm{id} \colon A \amalg A \to A$ is called the *fold map*. Dually, a *product* of two objects A and B in a category $\mathcal{C}$ is an object $A \times B$ together with two maps $\mathrm{pr}_1 \colon A \times B \to A$ and $\mathrm{pr}_2 \colon A \times B \to B$ such that, for every pair of maps $f \colon C \to A$ and $g \colon C \to B$, there exists a *unique* map

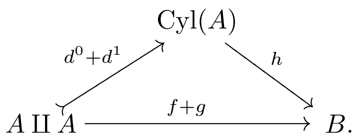
$$(f, g) \colon C \to A \times B$$

such that $f = \mathrm{pr}_1 \circ (f, g)$ and $g = \mathrm{pr}_2 \circ (f, g)$. If both $A \times B$ and $A \times' B$ are products of A and B, then the maps $(\mathrm{pr}_1, \mathrm{pr}_2) \colon A \times B \to A \times' B$ and $(\mathrm{pr}_1', \mathrm{pr}_2') \colon A \times' B \to A \times B$ are isomorphisms and each others inverses. The map $\Delta = (\mathrm{id}, \mathrm{id}) \colon A \to A \times A$ is called the *diagonal map.*

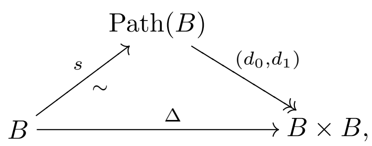
Definition Let $\mathcal{C}$ be a model category, and let $f \colon A \to B$ and $g \colon A \to B$ be two maps. A *cylinder object* for A is a commutative diagram



and a *left homotopy* from f to g is a commutative diagram



If a left homotopy from f to g exists, we say that f and g are *left homotopic* and write $f \overset{\ell}{\sim} g$. Dually, a *path object* for B is a commutative diagram



and a *right homotopy* from f to g is a commutative diagram

$$\begin{array}{ccc} & \mathrm{Path}(B) & \\ {}^{k}\nearrow & & \searrow^{(d_0,d_1)} \\ A & \xrightarrow{(f,g)} & B \times B. \end{array}$$

If a right homotopy from f to g exists, we say that f and g are *right homotopic* and write $f \overset{r}{\sim} g$. If both a left and a right homotopy from f to g exist, we say that f and g are homotopic and write $f \sim g$.

The homotopy relations $\overset{\ell}{\sim}$, $\overset{r}{\sim}$, and $\sim$ are relations on the set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ of maps from A to B in $\mathcal{C}$. But, in general, they are not equivalence relations. We write $\mathrm{Hom}_{\mathcal{C}}(A, B)/\overset{\ell}{\sim}$, $\mathrm{Hom}_{\mathcal{C}}(A, B)/\overset{r}{\sim}$, and $\mathrm{Hom}_{\mathcal{C}}(A, B)/\sim$ for the sets of equivalence classes for the equivalence relations on the set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ generated by the relations $\overset{\ell}{\sim}$, $\overset{r}{\sim}$, and $\sim$, respectively.