Cofibrantly generated model categories

Definition. Let $\mathcal{C}$ be a category such that all small colimits exist, and let

$$I = \{U \to V\}$$

be a class of maps in $\mathcal{C}$.

(i) An object X of $\mathcal{C}$ is *small relative to* I if for every sequence

$$Y_1 \xrightarrow{f_1} Y_2 \xrightarrow{f_2} Y_3 \xrightarrow{f_3} \dots$$

of maps in I, the canonical map

$$\operatorname*{colim}_i \operatorname{Hom}_{\mathcal{C}}(X, Y_i) \to \operatorname{Hom}_{\mathcal{C}}(X, \operatorname*{colim}_i Y_i)$$

is a bijection.

(ii) A map p is an *I-injective* if it has the right lifting property with respect to all maps in I.

(iii) A map i is an *I-cofibration* if it has the left lifting property with respect to all I-injective maps.

(iv) A map $f\colon A \to B$ is an *I-cellular* map if there exists a sequence of maps

$$A = B_0 \xrightarrow{i_0} B_1 \xrightarrow{i_1} B_2 \to \dots$$

together with push-out squares

$$\begin{array}{ccc} \amalg_\alpha U_\alpha & \longrightarrow & B_{m-1} \\ \downarrow & & \downarrow{\scriptstyle i_{m-1}} \\ \amalg_\alpha V_\alpha & \longrightarrow & B_m \end{array}$$

and maps $j_m\colon B_m \to B$ such that $j_0 = f$ and $j_m i_{m-1} = j_{m-1}$ and such that the induced map

$$j\colon \operatorname*{colim}_m B_m \to B$$

is an isomorphism.

Lemma. *(i) A retract of an I-injective map is an I-injective map.*

(ii) A retract of an I-cofibration is an I-cofibration.

(iii) Every I-cellular map is an I-cofibration.

Proposition (Small object argument). *Let $\mathcal{C}$ be a category in which all small colimits exist. Let I be a* set *of maps in $\mathcal{C}$ and assume that the domains of the maps in I are small relative to the class of I-cellular maps. Then every map f in $\mathcal{C}$ can be factored as $f = pi$, where i is an I-cellular map, and where p is an I-injective.*

Corollary. *Let $\mathcal{C}$ and I be as above. Then every I-cofibration is a retract of an I-cellular map.*

We say that a map is a *trivial fibration*, if is both a fibration and a weak equivalence, and that a map is a *trivial cofibration*, if it is both a cofibration and a weak equivalence.

Definition. A model category $\mathcal{C}$ is *cofibrantly generated* if there exists two *sets* of maps I and J such that the following (i)–(iv) hold.

(i) The domains of the maps in I are small relative to the I-cellular maps.

(ii) The domains of the maps in J are small relative to the J-cellular maps.

(iii) The fibrations are the J-injective maps.

(iv) The trivial fibrations are the I-injective maps.

We say that I is a set of *generating cofibrations* and that J is a set of *generating trivial cofibrations*.

Proposition. *Let $\mathcal{C}$ be a cofibrantly generated model category, let I be a set of generating cofibrations, and let J be a set of generating trivial cofibrations. Then the cofibrations are the I-cofibrations and the trivial cofibrations are the J-cofibrations.*

Theorem. *Let $\mathcal{C}$ be a category in which all small limits and colimits exist, and let $\mathcal{W}$ be a class of maps in $\mathcal{C}$ that is closed under retracts and satisfies the "two out of three" axiom (M2). Let I and J be two sets of maps in $\mathcal{C}$ and assume that the following (i)–(iv) holds.*

(i) The domains of the maps in I are small relative to the class of I-cellular maps. The domains of J are small relative to the class of J-cellular maps.

(ii) Every J-cofibration is both an I-cofibration and an element of $\mathcal{W}$.

(iii) Every I-injective is both a J-injective and an element of $\mathcal{W}$.

(iv) One of the following two conditions (a)–(b) hold:

(a) A map that is both an I-cofibration and an element of $\mathcal{W}$ is a J-cofibration.

(b) A map that is both a J-injective and an element of $\mathcal{W}$ is an I-injective.

Then $\mathcal{C}$ has a cofibrantly generated model structure, where $\mathcal{W}$ is the class of weak equivalences, where I is a set of generating cofibrations, and where J is a set of generating trivial cofibrations.