Limits and Colimits

By a *diagram* in a category $\mathcal{C}$, we mean a functor

$$\mathcal{X}\colon I \to \mathcal{C}$$

from a small category I to $\mathcal{C}$. Here a *small* category is a category whose class of objects form a set. The category of sets, for example, is not small because the class of all sets does not form a set. We say that I is the *index category* of $\mathcal{X}$ and that $\mathcal{X}$ is an I-*diagram*.

Definition. Let $\mathcal{X}$ be an I-diagram in $\mathcal{C}$. The *colimit* of $\mathcal{X}$ is an object

$$\operatorname*{colim}_I \mathcal{X}$$

of $\mathcal{C}$ that satisfies the following properties (i)–(ii):

(i) For every object i in I, there is a morphism in $\mathcal{C}$

$$j_i\colon \mathcal{X}(i) \to \operatorname*{colim}_I \mathcal{X},$$

and for every morphism $\varphi\colon i \to i'$ in I, $j_i = j_{i'} \circ \mathcal{X}(\varphi)$.

(ii) Given an object Y of $\mathcal{C}$ and, for every object i of I, a morphism in $\mathcal{C}$

$$f_i\colon \mathcal{X}(i) \to Y$$

such that, for every morphism $\varphi\colon i \to i'$ in I, $f_i = f_{i'} \circ \mathcal{X}(\varphi)$, there exists a *unique* morphism in $\mathcal{C}$

$$f\colon \operatorname*{colim}_I \mathcal{X} \to Y$$

such that, for all objects i in I, $f_i = f \circ j_i$.

We note that the colimit is well-defined up to canonical isomorphism. Indeed, if both $\operatorname{colim}_I \mathcal{X}$ and $\operatorname{colim}'_I \mathcal{X}$ satisfy (i) and (ii), then the unique morphisms

$$j'\colon \operatorname*{colim}_I \mathcal{X} \to \operatorname*{colim}_I{}' \mathcal{X}$$

and

$$j\colon \operatorname*{colim}_I{}' \mathcal{X} \to \operatorname*{colim}_I \mathcal{X}$$

are each others inverses.

Definition. Let $\mathcal{X}$ be an I-diagram in $\mathcal{C}$. The *limit* of $\mathcal{X}$ is an object

$$\lim_I \mathcal{X}$$

of $\mathcal{C}$ that satisfies the following properties (i)–(ii):

(i) For every object i in I, there is a morphism in $\mathcal{C}$

$$p_i\colon \lim_I \mathcal{X} \to \mathcal{X}(i),$$

and for every morphism $\varphi\colon i \to i'$ in I, $p_{i'} = \mathcal{X}(\varphi) \circ p_i$.

(ii) Given an object Z of $\mathcal{C}$ and, for every object i of I, a morphism in $\mathcal{C}$

$$g_i\colon Z \to \mathcal{X}(i)$$

such that, for every morphism $\varphi\colon i \to i'$ in I, $g_{i'} = \mathcal{X}(\varphi) \circ g_i$, there exists a *unique* morphism in $\mathcal{C}$

$$g\colon Z \to \lim_I \mathcal{X}$$

such that, for all objects i in I, $g_i = p_i \circ g$.

The limit is well-defined up to canonical isomorphism. In the litterature, the colimit is also called the *direct limit* and the *inductive limit* and the limit is called the *inverse limit* and the *projective limit*. It is also common to write $\varinjlim_I \mathcal{X}$ for the colimit and $\varprojlim_I \mathcal{X}$ for the limit.

EXAMPLE. Suppose that I has a terminal object ∞. Then, for every I-diagram $\mathcal{X}$, the object $\mathcal{X}(\infty)$ satisfies the conditions (i) and (ii), and hence,

$$\operatorname*{colim}_I \mathcal{X} = \mathcal{X}(\infty).$$

Similarly, if I has an initial object 0, then, for every I-diagram X, the object $X(0)$ satisfies the condition (i) and (ii). Hence, in this case,

$$\lim_I \mathcal{X} = \mathcal{X}(0).$$