Cofibrantly generated model categories

We first discuss limits and colimits of a diagram in a category $\mathscr{C}$. By a *diagram* in a category $\mathscr{C}$, we mean a functor

$$\mathscr{X} : I \to \mathscr{C}$$

from a small category I to $\mathscr{C}$. Here a *small* category is a category whose class of objects form a set. The category of sets, for example, is not small because the class of all sets does not form a set. We say that I is the *index category* of $\mathscr{X}$ and that $\mathscr{X}$ is an *I-diagram*.

DEFINITION. Let $\mathscr{X}$ be an I-diagram in $\mathscr{C}$. The *colimit* of $\mathscr{X}$ is an object

$$\operatorname*{colim}_I \mathscr{X}$$

of $\mathscr{C}$ that satisfies the following properties (i)–(ii):

(i) For every object i in I, there is a morphism in $\mathscr{C}$

$$j_i\colon \mathscr{X}(i) \to \operatorname*{colim}_I \mathscr{X},$$

and for every morphism $\varphi\colon i \to i'$ in I, $j_i = j_{i'} \circ \mathscr{X}(\varphi)$.

(ii) Given an object Y of $\mathscr{C}$ and, for every object i of I, a morphism in $\mathscr{C}$

$$f_i\colon \mathscr{X}(i) \to Y$$

such that, for every morphism $\varphi\colon i \to i'$ in I, $f_i = f_{i'} \circ \mathscr{X}(\varphi)$, there exists a *unique* morphism in $\mathscr{C}$

$$f\colon \operatorname*{colim}_I \mathscr{X} \to Y$$

such that, for all objects i in I, $f_i = f \circ j_i$.

We note that the colimit is well-defined up to canonical isomorphism. Indeed, if both $\operatorname{colim}_I \mathscr{X}$ and $\operatorname{colim}'_I \mathscr{X}$ satisfy (i) and (ii), then the unique morphisms

$$j'\colon \operatorname*{colim}_I \mathscr{X} \to \operatorname*{colim'}_I \mathscr{X}$$

and

$$j\colon \operatorname*{colim'}_I \mathscr{X} \to \operatorname*{colim}_I \mathscr{X}$$

are each others inverses.

DEFINITION. Let $\mathscr{X}$ be an I-diagram in $\mathscr{C}$. The *limit* of $\mathscr{X}$ is an object

$$\lim_I \mathscr{X}$$

of $\mathscr{C}$ that satisfies the following properties (i)–(ii):

(i) For every object i in I, there is a morphism in $\mathscr{C}$

$$p_i\colon \lim_I \mathscr{X} \to \mathscr{X}(i),$$

and for every morphism $\varphi\colon i \to i'$ in I, $p_{i'} = \mathscr{X}(\varphi) \circ p_i$.

(ii) Given an object Z of $\mathscr{C}$ and, for every object i of I, a morphism in $\mathscr{C}$

$$g_i\colon Z \to \mathscr{X}(i)$$

such that, for every morphism $\varphi\colon i \to i'$ in I, $g_{i'} = \mathscr{X}(\varphi) \circ g_i$, there exists a *unique* morphism in $\mathscr{C}$

$$g\colon Z \to \lim_I \mathscr{X}$$

such that, for all objects i in I, $g_i = p_i \circ g$.

The limit is well-defined up to canonical isomorphism. In the litterature, the colimit is also called the *direct limit* and the *inductive limit* and the limit is called the *inverse limit* and the *projective limit*. It is also common to write $\varinjlim_I \mathscr{X}$ for the colimit and $\varprojlim_I \mathscr{X}$ for the limit.

Example. Suppose that I has a terminal object ∞. Then, for every I-diagram $\mathscr{X}$, the object $\mathscr{X}(\infty)$ satisfies the conditions (i) and (ii), and hence,

$$\operatorname{colim}_I \mathscr{X} = \mathscr{X}(\infty).$$

Similarly, if I has an initial object 0, then, for every I-diagram X, the object $X(0)$ satisfies the condition (i) and (ii). Hence, in this case,

$$\lim_I \mathscr{X} = \mathscr{X}(0).$$

We next discuss the definition and properties of cofibrantly generated model categories. The small object argument shows that, in a cofibrantly generated model category, every object is weakly equivalent to an object that is built by attaching cells to each other. We begin with the precise definition of what this means.

Definition. Let $\mathscr{C}$ be a category such that all small colimits exist, and let

$$I = \{U \to V\}$$

be a class of maps in $\mathscr{C}$.

(i) An object X of $\mathscr{C}$ is *small relative to I* if for every sequence

$$Y_1 \xrightarrow{f_1} Y_2 \xrightarrow{f_2} Y_3 \xrightarrow{f_3} \dots$$

of maps in I, the canonical map

$$\operatorname{colim}_i \operatorname{Hom}_{\mathscr{C}}(X, Y_i) \to \operatorname{Hom}_{\mathscr{C}}(X, \operatorname{colim}_i Y_i)$$

is a bijection.

(ii) A map p is an *I-injective* if it has the right lifting property with respect to all maps in I.

(iii) A map i is an *I-cofibration* if it has the left lifting property with respect to all I-injective maps.

(iv) A map $f\colon A \to B$ is an *I-cellular* map if there exists a sequence of maps

$$A = B_0 \xrightarrow{i_0} B_1 \xrightarrow{i_1} B_2 \to \dots$$

together with push-out squares

$$\begin{array}{ccc} \amalg_\alpha U_\alpha & \longrightarrow & B_{m-1} \\ \downarrow & & \downarrow{\scriptstyle i_{m-1}} \\ \amalg_\alpha V_\alpha & \longrightarrow & B_m \end{array}$$

and maps $j_m\colon B_m \to B$ such that $j_0 = f$ and $j_m i_{m-1} = j_{m-1}$ and such that the induced map

$$j\colon \operatorname{colim}_m B_m \to B$$

is an isomorphism.

LEMMA. *(i) A retract of an I-injective map is an I-injective map.*

(ii) A retract of an I-cofibration is an I-cofibration.

(iii) Every I-cellular map is an I-cofibration.

PROPOSITION (Small object argument). *Let $\mathscr{C}$ be a category in which all small colimits exist. Let I be a* set *of maps in $\mathscr{C}$ and assume that the domains of the maps in I are small relative to the class of I-cellular maps. Then every map f in $\mathscr{C}$ can be factored as $f = pi$, where i is an I-cellular map, and where p is an I-injective.*

COROLLARY. *Let $\mathscr{C}$ and I be as above. Then every I-cofibration is a retract of an I-cellular map.*

We say that a map is a *trivial fibration*, if is both a fibration and a weak equivalence, and that a map is a *trivial cofibration*, if it is both a cofibration and a weak equivalence.

DEFINITION. A model category $\mathscr{C}$ is *cofibrantly generated* if there exists two *sets* of maps I and J such that the following (i)–(iv) hold.

(i) The domains of the maps in I are small relative to the I-cellular maps.

(ii) The domains of the maps in J are small relative to the J-cellular maps.

(iii) The fibrations are the J-injective maps.

(iv) The trivial fibrations are the I-injective maps.

We say that I is a set of *generating cofibrations* and that J is a set of *generating trivial cofibrations*.

PROPOSITION. *Let $\mathscr{C}$ be a cofibrantly generated model category, let I be a set of generating cofibrations, and let J be a set of generating trivial cofibrations. Then the cofibrations are the I-cofibrations and the trivial cofibrations are the J-cofibrations.*

THEOREM. *Let $\mathscr{C}$ be a category in which all small limits and colimits exist, and let $\mathscr{W}$ be a class of maps in $\mathscr{C}$ that is closed under retracts and satisfies the "two out of three" axiom (M2). Let I and J be two sets of maps in $\mathscr{C}$ and assume that the following (i)–(iv) holds.*

(i) The domains of the maps in I are small relative to the class of I-cellular maps. The domains of J are small relative to the class of J-cellular maps.

(ii) Every J-cofibration is both an I-cofibration and an element of $\mathscr{W}$.

(iii) Every I-injective is both a J-injective and an element of $\mathscr{W}$.

(iv) One of the following two conditions (a)–(b) hold:

(a) A map that is both an I-cofibration and an element of $\mathscr{W}$ is a J-cofibration.

(b) A map that is both a J-injective and an element of $\mathscr{W}$ is an I-injective.

Then $\mathscr{C}$ has a cofibrantly generated model structure, where $\mathscr{W}$ is the class of weak equivalences, where I is a set of generating cofibrations, and where J is a set of generating trivial cofibrations.