Algebraic Topology: Problem Set 1

Due: Tuesday, May 10.

Definition. Let $\mathscr{C}$ be a category.

(i) The *push-out* of two maps $f\colon A \to B$ and $g\colon A \to C$ in $\mathscr{C}$ is an object $B \sqcup_A C$ and two maps $g'\colon B \to B \sqcup_A C$ and $f'\colon C \to B \sqcup_A C$ such that $g' \circ f = f' \circ g$ and such that, for every pair of maps $h\colon B \to D$ and $k\colon C \to D$ with $h \circ f = k \circ g$, there exists a *unique* map $h + k\colon B \sqcup_A C \to D$ such that $(h+k) \circ g' = h$ and $(h+k) \circ f' = k$.

(ii) The map $f'\colon C \to B \sqcup_A C$ is called the *co-base change* of the map $f\colon A \to B$ along the map $g\colon A \to C$. The map $g'\colon B \to B \sqcup_A C$ is called the *co-base change* of the map $g\colon A \to C$ along the map $f\colon A \to B$.

(iii) The *pull-back* of two maps $f\colon B \to A$ and $g\colon C \to A$ in $\mathscr{C}$ is an object $B \times_A C$ and two maps $g'\colon B \times_A C \to B$ and $f'\colon B \times_A C \to C$ such that $f \circ g' = g \circ f'$ and such that, for every pair of maps $h\colon D \to B$ and $k\colon D \to C$ with $f \circ h = g \circ k$, there exists a *unique* map $(h,k)\colon D \to B \times_A C$ such that $g' \circ (h,k) = h$ and $f' \circ (h,k) = k$.

(iv) The map $f'\colon B \times_A C \to C$ is called the *base change* of the map $f\colon B \to A$ along the map $g\colon C \to A$. The map $g'\colon B \times_A C \to B$ is called the *base change* of the map $g\colon C \to A$ along the map $f\colon B \to A$.

Problem 1. Suppose that both

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ C & \xrightarrow{f'} & B \sqcup_A C \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g''} \\ C & \xrightarrow{f''} & B \sqcup'_A C \end{array}$$

are push-outs of the two maps $f\colon A \to B$ and $g\colon A \to C$. Show that the maps

$$f'' + g''\colon B \sqcup_A C \to B \sqcup'_A C$$
$$f' + g'\colon B \sqcup'_A C \to B \sqcup_A C$$

are inverse isomorphisms.

Problem 2. Suppose that $\mathscr{C}$ is a model category.

(i) Let $i\colon A \rightarrowtail B$ be a cofibration, and let $g\colon A \to C$ be any map. Show that the co-base change $i'\colon C \to B \sqcup_A C$ is a cofibration.

(ii) Let $i\colon A \overset{\sim}{\rightarrowtail} B$ be a trivial cofibration, and let $g\colon A \to C$ be any map. Show that the co-base change $i'\colon C \to B \sqcup_A C$ is a trivial cofibration.

(iii) Let $p\colon X \twoheadrightarrow Y$ be a fibration, and let $g\colon Z \to Y$ be any map. Show that the base change $p'\colon X \times_Y Z \to Z$ is a fibration.

(iv) Let $p\colon X \overset{\sim}{\twoheadrightarrow} Y$ be a trivial fibration, and let $g\colon Z \to Y$ be any map. Show that the base change $p'\colon X \times_Y Z \to Z$ is a trivial fibration.

Remark. It is *not* true, in general, that the co-base change of a weak equivalence is a weak equivalence or that the base change of a weak equivalence is a weak equivalence.