Algebraic Topology: Problem Set 2

Due: Tuesday, May 29.

DEFINITION. An *adjunction* from a category $\mathscr{C}$ to a category $\mathscr{D}$ is a triple (F, G, a) that consists of two functors $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{D} \to \mathscr{C}$ and a natural bijection

$$a = a_{(X,Y)}\colon \operatorname{Hom}_{\mathscr{D}}(F(X), Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathscr{C}}(X, G(Y)).$$

The functor F is said to be *a left adjoint* of the functor G, and the functor G is said to be *a right adjoint* of the functor F.

We emphasize that the bijection a must be *natural*. This means that we require the map $a = a_{(X,Y)}$ to be a natural transformation between the two functors

$$(X, Y) \mapsto \operatorname{Hom}_{\mathscr{D}}(F(X), Y)$$
$$(X, Y) \mapsto \operatorname{Hom}_{\mathscr{C}}(X, G(Y))$$

from the category $\mathscr{C}^{\mathrm{op}} \times \mathscr{D}$ to the category of sets. This, in turn, means that for every map $\varphi\colon X \to X'$ in $\mathscr{C}$, and for every map $\psi\colon Y \to Y'$ in $\mathscr{D}$, the diagram

$$\begin{array}{ccc}
\operatorname{Hom}_{\mathscr{D}}(F(X'), Y) & \xrightarrow{a_{(X',Y)}} & \operatorname{Hom}_{\mathscr{C}}(X', G(Y)) \\
\downarrow{\scriptstyle F(\varphi)^*} & & \downarrow{\scriptstyle \varphi^*} \\
\operatorname{Hom}_{\mathscr{D}}(F(X), Y) & \xrightarrow{a_{(X,Y)}} & \operatorname{Hom}_{\mathscr{C}}(X, G(Y)) \\
\downarrow{\scriptstyle \psi_*} & & \downarrow{\scriptstyle G(\psi)_*} \\
\operatorname{Hom}_{\mathscr{D}}(F(X), Y') & \xrightarrow{a_{(X,Y')}} & \operatorname{Hom}_{\mathscr{C}}(X, G(Y'))
\end{array}$$

must commute.

EXAMPLE. (i) Let $\mathscr{C}$ be the category of sets and maps, and let $\mathscr{D}$ be the category of abelian groups. Then we have the adjunction (F, G, a) from $\mathscr{C}$ to $\mathscr{D}$, where $F(X)$ is the free abelian group generated by the set X, and where $G(Y)$ is the underlying set of the abelian group Y. The bijection

$$a = a_{(X,Y)}\colon \operatorname{Hom}_{\mathscr{D}}(F(X), Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathscr{C}}(X, G(Y))$$

takes the group homomorphism $f\colon F(X) \to Y$ to the set map $a(f)\colon X \to G(Y)$ defined by $a(f)(x) = f(x)$. It is a bijection because the group homomorphism f is uniquely determined by its value on the basis X of $F(X)$.

(ii) Let $f\colon A \to B$ be a ring homomorphism, let $\mathscr{C}$ be the category of left A-modules and A-linear homomorphisms, and let $\mathscr{D}$ be the category of left B-modules and B-linear homomorphisms. We define an adjunction (F, G, a) from $\mathscr{C}$ to $\mathscr{D}$ as follows. If X is a left A-module, we define $F(X)$ to be the left B-module $B \otimes_A X$, and if Y is a left B-module, we define $G(Y)$ to be the left B-module Y considered as a left A-module via the map f, that is, we define $a \cdot y$ to be $f(a)y$. The bijection

$$a = a_{(X,Y)}\colon \operatorname{Hom}_{\mathscr{D}}(F(X), Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathscr{C}}(X, G(Y))$$

takes the B-linear map $g\colon F(X) \to Y$ to the A-linear map $a(g)\colon X \to G(Y)$ defined by $a(g)(x) = g(1 \otimes x)$. It is a bijection because g is uniquely determined by $a(g)$ by the formula $g(b \otimes x) = b \cdot g(1 \otimes x) = b \cdot a(g)(x)$.

(iii) Let $\mathscr{C}$ be a category, and let I be a small category. This means that the class of objects in I is a set. The category $\mathscr{C}^I$ of I-diagrams in $\mathscr{C}$ is defined as follows. The objects are all functors $\mathscr{X}: I \to \mathscr{C}$, and the set $\operatorname{Hom}_{\mathscr{C}^I}(\mathscr{X}, \mathscr{Y})$ of morphisms from an I-diagram $\mathscr{X}$ to an I-diagram $\mathscr{Y}$ is the set of all natural transformations from the functor $\mathscr{X}$ to the functor $\mathscr{Y}$. The *diagonal functor*

$$\Delta\colon \mathscr{C} \to \mathscr{C}^I$$

is defined by $\Delta(X)(i) = X$ and $\Delta(\varphi)(i) = \varphi$. Suppose that the colimit of every I-diagram in $\mathscr{C}$ exists. Then we can define an adjunction

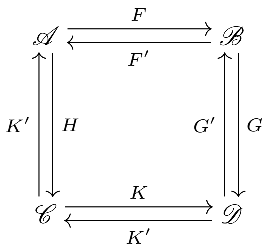
$$a\colon \operatorname{Hom}_{\mathscr{C}}(\operatorname*{colim}_I \mathscr{X}, Y) \to \operatorname{Hom}_{\mathscr{C}^I}(\mathscr{X}, \Delta(Y)),$$

where the map a takes the map $g\colon \operatorname{colim}_I \mathscr{X} \to Y$ to the natural transformation $a(g)\colon \mathscr{X} \to \Delta(Y)$, where $a(g)_i\colon \mathscr{X}(i) \to \Delta(Y)(i) = Y$ is the composite map

$$\mathscr{X}(i) \xrightarrow{j_i} \operatorname*{colim}_I \mathscr{X} \xrightarrow{a} Y.$$

In fact, the definition of the colimit of I-diagrams in $\mathscr{C}$ is equivalent to the definition of a left adjoint of the diagonal functor Δ. Similarly, the definition of the limit of I-diagrams in $\mathscr{C}$ is equivalent to the definition of a right adjoint of the diagonal functor Δ.

Problem 1. Consider the categories and functors



and assume that F (resp. G, resp. H, resp. K) is left adjoint to F' (resp. G', resp. H', resp. K'). Prove the following proposition:

(i) A natural transformation

$$\varphi_A\colon (G \circ F)(A) \to (K \circ H)(A)$$

determines and is determined by a natural transformation

$$\varphi'_D\colon (F' \circ G')(D) \to (H' \circ K')(D).$$

(ii) The natural transformation

$$\varphi_A\colon (G \circ F)(A) \to (K \circ H)(A)$$

is an isomorphism if and only if the corresponding natural transformation

$$\varphi'_D\colon (F' \circ G')(D) \to (H' \circ K')(D)$$

is an isomorphism.

The proposition of Problem 1 is extremely useful. For example, the following problem would be very tedious to solve directly.

PROBLEM 2. Let $f\colon A \to B$ be a ring homomorphism, and let I be a small category. Prove that, for every I-diagram of left A-modules $\mathscr{X}$, the canonical map

$$\varphi_{\mathscr{X}}\colon B \otimes_A \operatorname*{colim}_I \mathscr{X} \to \operatorname*{colim}_I (B \otimes_A \mathscr{X})$$

is an isomorphism. (Hint: Use Problem 1.)