The Topologist's Sine Curve

We consider the subspace $X = X' \cup X''$ of $\mathbb{R}^2$, where

$$X' = \{(0, y) \in \mathbb{R}^2 \mid -1 \leqslant y \leqslant 1\},$$
$$X'' = \{(x, \sin\frac{1}{x}) \in \mathbb{R}^2 \mid 0 < x \leqslant \frac{1}{\pi}\}.$$

We will prove below that the map

$$f\colon S^0 \to X$$

defined by $f(-1) = (0, 0)$ and $f(1) = (1/\pi, 0)$ is a weak equivalence but not a homotopy equivalence. But first we discuss some of the basic topological properties of the space X. For basic general topology, see e.g. J. R. Munkres, *Topology, Second Edition*, Prentice Hall 2000.

We recall that a space Y is defined to be *connected* if the only subsets $A \subset Y$ that are both open and closed are $A = \emptyset$ and $A = Y$. A subset $A \subset Y$ is defined to be *dense* if the smallest closed subset of Y that contains A is Y. If $A \subset Y$ is a dense subset, and if A is a connected space, then Y is connected. We next recall that a space Y is defined to be *path-connected* if every two points of Y can be joined by a curve, that is, if for every pair (y, y') of points of Y, there exists a continuous map $\sigma\colon [0, 1] \to Y$ such that $\sigma(0) = y$ and $\sigma(1) = y'$. A path-connected space is always connected, but the converse is not always true. If $f\colon Y \to Z$ is a continuous map, and if Y is connected (resp. path-connected), then the image $f(Y) \subset Z$ is again connected (resp. path-connected). We note that a non-empty space Y is path-connected if and only if, for some choice of base-point $y_0 \in Y$, the set $\pi_0(Y, y_0)$ consists of a single element.

The subspaces X' and X'' of X are both path-connected, and hence, connected. Moreover, $X'' \subset X$ is dense, and therefore, X is connected. However, X is not path-connected. To see this, suppose, conversely, that there exists a continuous map $\sigma\colon [0, 1] \to X$ such that $\sigma(0) = (0, 0)$ and $\sigma(1) = (1/\pi, 0)$. Since $X' \subset X$ is closed, and since σ is continuous, the subset

$$A = \sigma^{-1}(X') \subset [0, 1]$$

is a closed subset. Therefore, the supremum $a = \sup A$ is an element of A. We can assume without loss of generality that $a = 0$. Then we have $\sigma(0) = (0, 0)$, and $\sigma(t) = (x(t), y(t))$ with $x(t) > 0$ and $y(t) = \sin(1/x(t))$, for $t > 0$. Now, for every positive integer n, we choose $0 < u_n < x(1/n)$ such that $\sin(1/u_n) = (-1)^n$. Then, by the mean value theorem, there exists $0 < t_n < 1/n$ such that $x(t_n) = u_n$. By construction, the sequence $\{t_n\}$ converges to 0, but $\{\sigma(t_n)\}$ does not converge. This contracdicts the continuity of σ at 0. Therefore, X is not path-connected.

We conclude from the above that the induced map

$$f_*\colon \pi_0(S^0, s_0) \to \pi_0(X, f(s_0))$$

is a bijection. We proceed to show that the same is true for the induced map of higher homotopy groups. Let n be a positive integer, and let $x_0 \in X$ be a point. We first show that the set $\pi_n(X, x_0)$ consists of a single element, or equivalently, that for every continuous base-point preserving map

$$\varphi\colon (S^n, s_0) \to (X, x_0),$$

there exists a base-point preserving homotopy from φ to the constant base-point preserving map. Since n is a positive integer, the sphere S^n is path-connected. Therefore, the image $\varphi(S^n) \subset X$ is a path-connected space, and hence, either $\varphi(S^n) \subset X'$ or $\varphi(S^n) \subset X''$. Suppose that $\varphi(S^n) \subset X'$. Then the map

$$F\colon S^n \times [0,1] \to X$$

defined by

$$F(s,t) = t\varphi(s) + (1-t)x_0$$

is a base-point preserving homotopy from φ to the constant map with value x_0. Similarly, suppose that $\varphi(S^n) \subset X''$ such that $\varphi(s) = (x(s), y(s))$ with $y(s) = \sin(1/x(s))$. Then the map

$$F\colon S^n \times [0,1] \to X$$

defined by the formula

$$F(s,t) = \left(tx(s) + (1-t)x(s_0), \sin\left(\frac{1}{tx(s) + (1-t)x(s_0)} \right) \right)$$

is a base-point preserving homotopy from φ to the constant map with value x_0. It follows that for every positive integer n, the induced map

$$f_*\colon \pi_n(S^0, s_0) \to \pi_n(X, f(s_0))$$

is a bijection. Indeed, the domain and range of f_* both have exactly one element.

Finally, we show that the map f is not a homotopy equivalence. Since X is a non-empty connected space, the image of any continuous map

$$g\colon X \to S^0$$

is a non-empty connected subspace $A \subset S^0$. But there only two such subspaces: $A = \{-1\}$ and $A = \{1\}$. It follows that g is a constant map, so $g \circ f$ cannot be homotopic to the identity map of S^0. Hence, f is not a homotopy equivalence.