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A <u>homotopy</u> from the continuous map $f : X \to Y$ to the continuous map $g : X \to Y$ is a continuous map

$$F : X \times [0,1] \longrightarrow Y$$

such that

$$F(x,0) = f(x) \qquad \text{for all } x \in X$$

$$F(x,1) = g(x) \qquad \text{for all } x \in X.$$

If there exists a homotopy from f to g, we say that f is homotopic to g and write $f \sim g$. This is an <u>equivalence</u> relation:

$$f \sim f$$

$$f \sim g \Rightarrow g \sim f$$

$$f \sim g \text{ and } g \sim h \Rightarrow f \sim h.$$

The map $f : X \to Y$ is called a <u>homotopy</u> <u>equivalence</u>, if there exists a map $g : Y \to X$ such that

$$f \circ g \sim \mathrm{id}_Y$$

$$g \circ f \sim \mathrm{id}_X$$

If there exists a homotopy equivalence $f : X \to Y$, we say that X and Y are <u>homotopy equivalent</u> and write $X \simeq Y$.

Let $*$ a space with one point. If $X \simeq *$, we say that X is a <u>contractible</u> space.

<u>Ex</u> $\mathbb{R}^n$ is contractible.

$\mathbb{R}^n \setminus \{0\} \simeq S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$. //

We will associate a space $B\mathcal{C}$ to every category $\mathcal{C}$. First, we introduce simplicial sets.

Let Δ be the category whose objects are the ordered sets

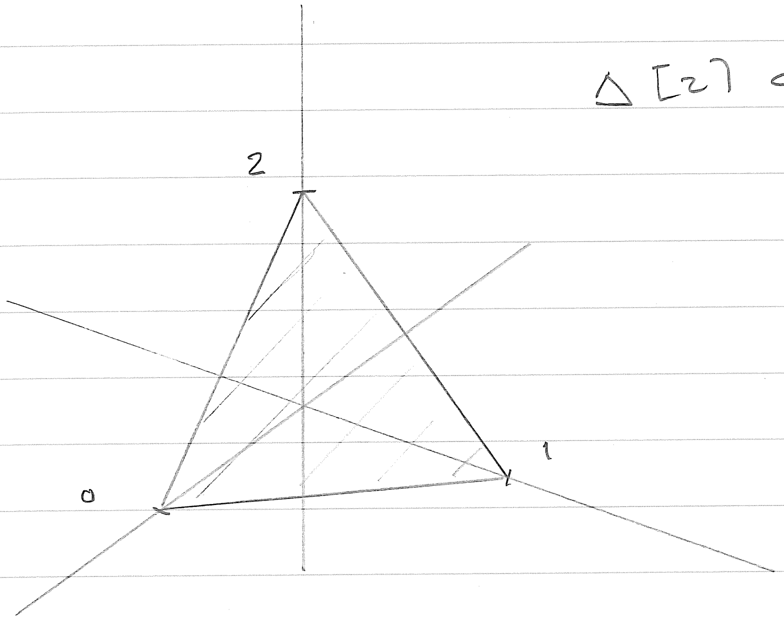
$$[n] = \{0 < 1 < 2 < \dots < n\} \quad (n \geq 0)$$

and whose morphisms are all

non-decreasing maps. We let $\mathbb{R}[n]$ be the $\mathbb{R}$-vector space with basis $[n]$ and define

$$\Delta[n] \subset \mathbb{R}[n]$$

to be the convex hull of the set of basis elements $[n] \subset \mathbb{R}[n]$:



So every $x \in \Delta[n]$ can be written uniquely as

$$z = \sum_{i \in [n]} a_i \cdot i$$

where $a_i \in [0,1]$ and

$$\sum_{i \in [n]} a_i = 1.$$

The map $\theta : [m] \to [n]$ gives rise to a continuous map

$$\theta_* : \Delta[m] \longrightarrow \Delta[n]$$

defined by

$$\begin{aligned}\theta_*\Big(\sum_{i\in[m]} a_i \cdot i\Big) &= \sum_{i\in[m]} a_i \cdot \theta(i) \\ &= \sum_{j\in[n]} \Big(\sum_{i\in\theta^{-1}(j)} a_i\Big)\cdot j\end{aligned}$$

This defines a functor

$$\Delta[-] : \boldsymbol{\Delta} \longrightarrow \mathcal{T}$$

from $\boldsymbol{\Delta}$ to the category $\mathcal{T}$ of topological spaces and continuous maps.

$$\begin{tikzcd}
{[m]} \arrow[r, mapsto] & \Delta[m] \\
\downarrow\theta \arrow[r, mapsto] & \downarrow\theta_* \\
{[n]} \arrow[r, mapsto] & \Delta[n]
\end{tikzcd}$$

A simplicial set is a functor

$$\begin{tikzcd}
X[-] : \Delta^{op} \arrow[r] & Sets \\
{[m]} \arrow[r, mapsto] \arrow[d, "\theta"] & X[m] \\
{[n]} \arrow[r, mapsto] & X[n] \arrow[u, "\theta^*"']
\end{tikzcd}$$

This is a "recipe" for building a space out of simplices: We define the geometric realization of $X[-]$ to be the space

$$|X[-]| := \left(\coprod_{n \geq 0} X[n] \times \Delta[n] \right) / \sim$$

where $\sim$ is the equivalence relation generated by the relation that, for all $\theta : [m] \to [n]$ in Δ, all $x \in X[n]$, and all $z \in \Delta[m]$, identifies

$$(x, \theta_*(z)) \in X[n] \times \Delta[n]$$

$$\wr$$

$$(\theta^*(x), z) \in X[m] \times \Delta[m] .$$

Here $X[n]$ is given the discrete topology, $X[n] \times \Delta[n]$ the product topology, and $|X[-]|$ the quotient topology. We note that

$$X[n] \times \Delta[n] = \coprod_{x \in X[n]} \Delta[n].$$

The geometric realization was defined by John Milnor. We will consider it in more detail next time.

We now define the space of a category $\mathcal{C}$. We assume that $\mathcal{C}$ is <u>small</u> in the sense that the objects of $\mathcal{C}$ form a set. We can view $[n]$ as the category

$$0 \to 1 \to 2 \to \cdots \to n.$$

Then a non-decreasing map

$$\theta : [m] \to [n]$$

is the same as a functor

$\theta : [m] \longrightarrow [n]$.

The <u>nerve</u> of $\mathcal{C}$ is the simplicial set

$$N\mathcal{C}[-] : \Delta^{op} \longrightarrow \text{Sets}$$

defined by

$$\begin{tikzcd}
{[m]} \arrow[r, mapsto] \arrow[d, "\theta"] & \text{Funct}([m], \mathcal{C}) \\
{[n]} \arrow[r, mapsto] & \text{Funct}([n], \mathcal{C}) \arrow[u, "\theta^*"']
\end{tikzcd}$$

where $\text{Funct}([n], \mathcal{C})$ is the set of all functors $\alpha : [n] \longrightarrow \mathcal{C}$ and where θ^* is defined by

$$\theta^*([n] \xrightarrow{\alpha} \mathcal{C})$$

$$= [m] \xrightarrow{\theta} [n] \xrightarrow{\alpha} \mathcal{C}.$$

The geometric realization

$$B\mathcal{C} := |N\mathcal{C}[-]|$$

is the space of the category $\mathcal{C}$. A functor $f: \mathcal{C} \to \mathcal{D}$ gives rise to the maps

$$Nf[n] : N\mathcal{C}[n] \to N\mathcal{D}[n]$$

defined by

$$([n] \xrightarrow{\alpha} \mathcal{C}) \mapsto ([n] \xrightarrow{\alpha} \mathcal{C} \xrightarrow{f} \mathcal{D}).$$

The maps $Nf[n]$ $(n \geq 0)$ form a natural transformation

$$Nf[-] : N\mathcal{C}[-] \to N\mathcal{D}[-]$$

which means that, for all $\theta : [m] \to [n]$ in Δ, the diagram

$$\begin{tikzcd}
N\mathcal{C}[m] \arrow[r, "{Nf[m]}"] & N\mathcal{D}[m] \\
N\mathcal{C}[n] \arrow[r, "{Nf[n]}"] \arrow[u, "\theta^*"'] & N\mathcal{D}[n] \arrow[u, "\theta^*"']
\end{tikzcd}$$

commutes. The natural transformation $Nf[-]$, in turn, induces a continuous map

$$Bf : B\mathcal{C} \longrightarrow B\mathcal{D}$$

that takes the class of

$$(\alpha, z) \in N\mathcal{C}[n] \times \Delta[n]$$

to the class of

$$(f \circ \alpha, z) \in N\mathcal{D}[n] \times \Delta[n].$$

Next, let φ be a natural transformation from the functor $f : \mathcal{C} \to \mathcal{D}$ to the functor $g : \mathcal{C} \to \mathcal{D}$. This means that, for every object $c \in \operatorname{ob} \mathcal{C}$ we have a morphism $\varphi_c : f(c) \to g(c)$ in $\mathcal{D}$ such that, for every morphism $\alpha : c \to c'$ in $\mathcal{C}$, the following diagram in $\mathcal{D}$ commutes:

$$\begin{tikzcd}
f(c) \arrow[r] \arrow[d, "f(\alpha)"] & g(c) \arrow[d, "g(\alpha)"] \\
f(c') \arrow[r, "\varphi_{c'}"] & g(c')
\end{tikzcd}$$

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let $\mathcal{E} \times [1]$ be the product category, where the objects are

$$ob(\mathcal{E} \times [1]) = ob\,\mathcal{E} \times ob[1] ,$$

and where a morphism from (c, i) to (c', i') is a pair (α, β) of a morphism $\alpha : c \to c'$ in $\mathcal{E}$ and a morphism $\beta : i \to i'$ in $[1]$. The natural transformation φ gives rise to a functor

$$F : \mathcal{E} \times [1] \longrightarrow \mathcal{D}$$

defined by

$$\begin{tikzcd}
F(c, 0) \arrow[d, "{F(id_c, 0 \to 1)}"] & = & f(c) \arrow[d, "\varphi_c"] \\
F(c, 1) & = & g(c)
\end{tikzcd}$$

$$\begin{tikzcd}
F(c, 0) \arrow[r, equal] \arrow[d, "{F(\alpha, id_0)}"] & f(c) \arrow[d, "f(\alpha)"] \\
F(c', 0) \arrow[r, equal] & f(c')
\end{tikzcd}$$

$$\begin{tikzcd}
F(c, 1) \arrow[r, equal] \arrow[d, "{F(\alpha, id_1)}"] & g(c) \arrow[d, "g(\alpha)"] \\
F(c', 1) \arrow[r, equal] & g(c')
\end{tikzcd}$$

Conversely, the functor

$$F : \mathcal{C} \times [1] \longrightarrow \mathcal{D}$$

determines two functors,

$$f, g : \mathcal{C} \longrightarrow \mathcal{D}$$

$$f(c) = F(c, 0)$$

$$g(c) = F(c, 1),$$

and a natural transformation φ from f to g,

$$\begin{tikzcd}
f(c) \arrow[d, "\varphi_c"] & = & F(c, 0) \arrow[d, "F(id_c, 0\to 1)"] \\
g(c) & = & F(c, 1)
\end{tikzcd}$$

So the natural transformation φ determines the functor F which, in turn, induces a continuous map

$$BF : B(\mathcal{C} \times [1]) \to B\mathcal{D}.$$

The projection functors

$$pr_1 : \mathcal{C} \times [1] \to \mathcal{C}$$

$$(c, i) \mapsto c$$

$$pr_2 : \mathcal{C} \times [1] \to [1]$$

$$(c, i) \mapsto i$$

induce continuous maps

$$Bpr_1 : B(\mathcal{C} \times [1]) \to B\mathcal{C}$$

$$B\,pr_2 : B(\mathcal{E} \times [1]) \longrightarrow B[1],$$

and hence, a continuous map

$$(B\,pr_1, B\,pr_2) : B(\mathcal{E} \times [1]) \longrightarrow B\mathcal{E} \times B[1].$$

We will prove next time that this map is a homeomorphism. There is also a continuous map

$$k : \Delta[1] \longrightarrow B[1]$$

defined by

$$k(z) = \begin{pmatrix} \text{class of} \\ \\ (\iota, z) \in N[1][1] \times \Delta[1] \end{pmatrix}$$

where $\iota : [1] \to [1]$ is the identity functor. It is clear that this map is surjective: If

$$(\alpha, z) \in N[1][n] \times \Delta[n]$$

then $\alpha = \alpha^* \iota$, so

$$(\alpha, z) = (\alpha^* \iota, z) \sim (\iota, \alpha_* z).$$

We will show next time that the map is also injective. Since $\Delta[1]$ is a compact Hausdorff space, it follows that this map is a homeomorphism. In any case, the natural transformation φ from $f : \mathcal{C} \to \mathcal{D}$ to $g : \mathcal{C} \to \mathcal{D}$ gives rise to the continuous map

$$B\varphi : B\mathcal{C} \times \Delta[1] \longrightarrow B\mathcal{D}$$

defined by the composition

$$B\mathcal{C} \times \Delta[1] \xrightarrow{id \times h} B\mathcal{C} \times B[1]$$

$$\xrightarrow{(Bpr_1, Bpr_2)^{-1}} B(\mathcal{C} \times [1]) \xrightarrow{BF} B\mathcal{D}.$$

This map is a homotopy from Bf to Bg. Hence, we have

$\mathcal{C}$: category $\longmapsto$ $B\mathcal{C}$ topological space

$f : \mathcal{C} \to \mathcal{D}$ functor $\longmapsto$ $Bf : B\mathcal{C} \to B\mathcal{D}$ continuous map

$\varphi : f \to g$ natural transformation $\longmapsto$ $B\varphi : B\mathcal{C} \times \Delta[1] \to B\mathcal{D}$ homotopy.

Example Suppose that $\mathcal{C}$ has a terminal object ∞. This means that, for every object $c \in ob\,\mathcal{C}$, there is a unique morphism

$$\varphi_c : c \longrightarrow \infty$$

Let $*$ be the category with one object $*$ and one morphism id_*. We have functors

$$f : \mathcal{C} \longrightarrow * \ , \quad f(c) = *$$

$$g : * \longrightarrow \mathcal{C} \ , \quad g(*) = \infty \ .$$

The composition $f \circ g$ is the identity functor, and we have a natural transformation from the identity functor of $\mathcal{C}$ to $g \circ f$ given by the unique morphism

$$\varphi_c : c \longrightarrow \infty = (g \circ f)(c) \ .$$

It follows that

$$Bf : B\mathcal{C} \longrightarrow B* = *$$

is a homotopy equivalence. So BE is a contractible space.