We will first present Quillen's small object argument which is used to prove the factorizations M5. Let

$$I = \{U \to V\}$$

be a class of maps in the category $\mathcal{C}$. Then the object $X$ of $\mathcal{C}$ is <u>small</u> relative to $I$ if, for every sequence of maps in $I$,

$$Y_1 \to Y_2 \to Y_3 \to \cdots,$$

the canonical map

$$\operatorname*{colim}_i \operatorname{Hom}_{\mathcal{C}}(X, Y_i) \to \operatorname{Hom}_{\mathcal{C}}(X, \operatorname*{colim}_i Y_i)$$

is a bijection. Suppose that $\mathcal{C}$ is a model category. Then we say that $\mathcal{C}$ is <u>cofibrantly generated</u> if there are two <u>sets</u> of maps in $\mathcal{C}$, $I$ and $J$, such that the following (i) - (iv) hold:

(i) The domains of the maps in $I$ are small relative to the class of relative $I$-cell complexes.

(ii) The domains of the maps in $J$ are small relative to the class of relative $J$-cell complexes.

(iii) The fibrations are the maps which have RLP with respect to the maps in $J$.

(iv) The trivial fibrations are the maps which have RLP with respect to the maps in $I$.

The set $I$ (resp. $J$) is called the set of <u>generating cofibrations</u> (resp. <u>generating trivial cofibrations</u>) because it follows from (i)-(iv) that

(v) The cofibrations are the $I$-cofibrations.

(vi) The trivial cofibrations are the $J$-cofibrations.

The Serre model structure on $\mathcal{K}$ is cofibrantly generated with generating cofibrations and generating trivial cofibrations

$$I = \{ \partial D^n \rightarrowtail D^n \mid n \geq 0 \}$$

$$J = \{ D^n \overset{\sim}{\rightarrowtail} D^n \times [0,1] \mid n \geq 0 \}.$$

It is not difficult to verify (i) and (ii), and (iii) is the definition of a Serre fibration. It takes more work to prove (iv); see Hovey.

<u>Prop</u> (Small object argument) Let $\mathcal{C}$ be a category in which all small colimits exist. Let $I$ be a set of maps in $\mathcal{C}$ and assume that the domains of maps in $I$ are small relative to the class of relative $I$-cell complexes. Then every map $f$ in $\mathcal{C}$ can be factored as the composition

$$f = p \circ i$$

where $i$ is a relative $I$-cell complex and $p$ has RLP with respect to the maps in $I$. This factorization is functorial.

Proof The last sentence means that $p$ and $i$ depend functorially on the map $f$. This will be clear from the proof. We construct the factorization

$$\begin{tikzcd} & Z \arrow[dr, "p"] & \\ X \arrow[ur, "i"] \arrow[rr, "f"] & & Y \end{tikzcd}$$

as follows: By induction on $n \geq 0$, we construct maps

$$X = Z_0 \xrightarrow{i_1} Z_1 \xrightarrow{i_2} Z_2 \xrightarrow{i_3} \cdots$$

and

$$p_n : Z_n \longrightarrow Y$$

with $p_0 = f$ and $p_n \circ i_n = p_{n-1}$. Let $Z_0 = X$ and $p_0 = f$. Suppose, inductively, that $Z_m$, $i_m$, and $p_m$ have been constructed, for $m < n$. We let $\mathcal{D}_n$ be the set of all diagrams of the form

$$\alpha: \quad \begin{array}{ccc} U_\alpha & \xrightarrow{h_\alpha} & Z_{n-1} \\ \Big\downarrow g_\alpha & & \Big\downarrow p_{n-1} \\ V_\alpha & \xrightarrow{k_\alpha} & Y \end{array}$$

with $g_\alpha$ in $I$. Since $I$ is a set (as opposed to a class), $\mathcal{D}_n$ is also a set. Hence, we may form the following push-out, which defines $Z_n$, $i_n$, and $p_n$:

$$\begin{tikzcd} \coprod_{\alpha \in \mathcal{D}_n} U_\alpha \arrow[r, "\Sigma h_\alpha"] \arrow[d, "\coprod g_\alpha"] & Z_{n-1} \arrow[d, "i_n"] \arrow[ddr, bend left, "p_{n-1}"] & \\ \coprod_{\alpha \in \mathcal{D}_n} V_\alpha \arrow[r] \arrow[drr, bend right, "\Sigma k_\alpha"'] & Z_n \arrow[dr, "p_n"] & \\ & & Y \end{tikzcd}$$

We then define

$$Z = \operatorname{colim}_n Z_n$$

and define $i$ and $p$ to be the maps given by the $i_n$ and $p_n$. Then $i$ is a relative $I$-cell complex by construction, so it remains only to prove that $p$ has RLP with respect to maps in $I$. Let

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "g"'] & Z \arrow[d, "p"] \\ V \arrow[r, "k"] \arrow[ur, dashed, "\tilde{k}"] & Y \end{tikzcd}$$

be a commutative diagram with $g$ in $I$. We wish to show that there exists a map $\tilde{k}: V \to Z$ such that $k = p \circ \tilde{k}$ and $h = \tilde{k} \circ g$. Since $U$ is small with respect to the class of relative $I$-cell complexes, the map $h$ factors as

$$\begin{tikzcd} U \arrow[r, "h'"] \arrow[rr, bend right, "h"'] & Z_{n-1} \arrow[r, "j_{n-1}"] & Z \end{tikzcd}$$

for some $n$. But then the diagram

$$\begin{tikzcd} U \arrow[r, "h'"] \arrow[d, "q"] & Z_{n-1} \arrow[d, "p_{n-1}"] \\ V \arrow[r, "k"] & Y \end{tikzcd}$$

is an element of $D_n$. Hence, we have

$$\begin{tikzcd} U \arrow[r, "h'"] \arrow[d, "q"] & Z_{n-1} \arrow[d, "i_n"] \arrow[dr, bend left, "p_{n-1}"] & \\ V \arrow[r, "k'"] \arrow[rr, bend right, "k"'] & Z_n \arrow[r, "p_n"] & Y \end{tikzcd}$$

so

$$\begin{tikzcd} U \arrow[r, "h'"] \arrow[d, "q"] \arrow[rr, bend left, "h"] & Z_{n-1} \arrow[r, "j_{n-1}"] \arrow[d, "i_n"] & Z \arrow[d, "p"] \\ V \arrow[r, "k'"] \arrow[rr, bend right, "k"'] & Z_n \arrow[r, "p_n"] \arrow[ur, "j_n"] & Y \end{tikzcd}$$

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shows that $\tilde{k} = j_n \circ k'$ is the desired lifting. //

We will next construct a model structure on the category $\mathcal{K}^{\Delta^{op}}$ of simplicial spaces. Let $\Delta^+$ and $\Delta^-$ be the subcategories of $\Delta$ that both have all objects and where $\theta\colon [m] \to [n]$ is in $\Delta^+$ (resp. $\Delta^-$) if and only if $\theta$ is surjective (resp. injective). Then every map $\theta$ in $\Delta$ can be factored uniquely as the composition

$$\theta = \theta^- \circ \theta^+$$

of a map $\theta^+$ in $\Delta^+$ and a map $\theta^-$ in $\Delta^-$. Let $X[-]$ be a simplicial space, and let $n \geq 0$. We define the $n$'th latching space of $X[-]$ by

$$L_n X := \operatorname*{colim}_{\substack{\theta\colon [n] \to [p] \\ p < n}} X[p] \xleftarrow{\cong} \operatorname*{colim}_{\substack{\theta^+\colon [n] \twoheadrightarrow [p] \\ p < n}} X[p]$$

and the n'th <u>matching space</u> by

$$M_n X = \lim_{\substack{\eta : [m] \to [n] \\ m < n}} X[m]$$

$$\xrightarrow{\cong} \lim_{\substack{\bar{\eta} : [m] \hookrightarrow [n] \\ m < n}} X[m] .$$

We have canonical maps

$$L_n X \longrightarrow X[n] \longrightarrow M_n X .$$

The composition of these two maps is the canonical map

$$\begin{tikzcd}
\operatorname{colim}\limits_{\substack{\theta : [n] \to [p] \\ p < n}} X[p] \arrow[rr] & & \lim\limits_{\substack{\eta : [m] \to [n] \\ m < n}} X[m] \arrow[d, "pr_\eta"] \\
X[p] \arrow[u, "in_\theta"] \arrow[rr, "(\theta \circ \eta)^*"] & & X[m]
\end{tikzcd}$$

Let $\psi = \theta \circ \eta$. Then we have the unique factorization

$$[m] \xrightarrow{\eta^+} [k] \xrightarrow{\eta^-} [p]$$

where also $k < n$. Hence, to define the canonical map

$$L_n X \longrightarrow M_n X,$$

we need only know the restriction $X_{<n}[-]$ of $X[-]$ to the full subcategory $\Delta_{<n}$ of $\Delta$ with objects $[k]$, $0 \leq k < n$.

<u>Lemma</u> Let $X_{<n}[-]$ be a functor from $\Delta^{op}_{<n}$ to $\mathcal{K}$. Then the following are equivalent:

(i) A functor $X_{<n+1}[-]$ from $\Delta^{op}_{<n+1}$ to $\mathcal{K}$ whose restriction to $\Delta^{op}_{<n}$ is $X_{<n}[-]$.

(ii) A space $X[n]$ and a factorization

$$L_n X_{<n} \longrightarrow X[n] \longrightarrow M_n X_{<n}$$

of the canonical map.

<u>Proof</u> We have already seen that (i)

implies (ii). So assume (ii). We define

$$X_{<n+1}[m] = \begin{cases} X_{<n}[m] & (m < n) \\ X[n] & (m = n). \end{cases}$$

Let $\theta : [k] \to [p]$ be a non-identity map in $\Delta_{<n+1}$. We factor $\theta$ as

$$[k] \overset{\theta^+}{\twoheadrightarrow} [m] \xrightarrow{\theta^-} [p] .$$

Then $m < n$, since $\theta \neq \mathrm{id}_{[n]}$. We then define

$$\theta^* : X_{<n+1}[p] \to X_{<n+1}[k]$$

to be the composition

$$X_{<n+1}[p] \to M_p X_{<n+1} \xrightarrow{\mathrm{pr}_{\theta^-}} X_{<n+1}[m]$$
$$\xrightarrow{\mathrm{in}_{\theta^+}} L_k X_{<n+1} \to X_{<n+1}[k]$$

where the first and last map are either the canonical maps or the maps given in (ii). //

<u>Thm</u> The category $\mathcal{K}^{\Delta^{op}}$ of simplicial spaces has a model structure where the map $f : X[-] \to Y[-]$ is a weak equivalence if and only if the maps $f : X[n] \to Y[n]$ are weak equivalences in $\mathcal{K}$, for all $n \geq 0$; a (trivial) cofibration if and only if, the induced maps

$$X[n] \amalg_{L_n X} L_n Y \longrightarrow Y[n]$$

are (trivial) cofibrations in $\mathcal{K}$, for all $n \geq 0$; a (trivial) fibration if and only if the induced maps

$$X[n] \longrightarrow Y[n] \times_{M_n Y} M_n X$$

are (trivial) fibrations in $\mathcal{K}$, for all $n \geq 0$.

<u>Proof</u> See Hovey, pages 120–126. The proof uses the lemma above. //

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We next prove the following result which is known as Quillen's SM7.

<u>Prop</u> The following equivalent statements hold in $\mathcal{K}$:

(i) Let $i : A \rightarrowtail X$ and $j : B \rightarrowtail Y$ be two cofibrations. Then the induced map

$$A \times Y \amalg_{A \times B} X \times B \longrightarrow X \times Y$$

is a cofibration which is trivial if $i$ or $j$ is trivial.

(ii) Let $j : B \rightarrowtail Y$ and $p : Z \twoheadrightarrow W$ be a cofibration and a fibration. Then the induced map

$$\underline{\mathrm{Hom}}(Y, Z) \longrightarrow \underline{\mathrm{Hom}}(Y, W) \times_{\underline{\mathrm{Hom}}(B, W)} \underline{\mathrm{Hom}}(B, Z)$$

is a fibration which is trivial if $j$ or $p$ is trivial.

<u>Proof</u> We first argue that (i) and (ii) are equivalent. Recall that a map is a cofibration (resp. trivial cofibra-

tion) if and only if it has LLP with respect to trivial fibrations (resp. fibrations). Similarly, a map is a fibration (resp. trivial fibration) if and only if it has RLP with respect to trivial cofibrations (resp. cofibrations). Now, by adjointness, the commutative diagram

$$
\begin{tikzcd}
A\times Y \sqcup_{A\times B} X\times B \arrow[r] \arrow[d] & Z \arrow[d] \\
X\times Y \arrow[r] \arrow[ur, dashed] & W
\end{tikzcd}
$$

determines and is determined by the commutative diagram

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(Y, Z) \arrow[d] \\
X \arrow[r] \arrow[ur, dashed] & \underline{\mathrm{Hom}}(Y, W) \times_{\underline{\mathrm{Hom}}(B, W)} \underline{\mathrm{Hom}}(B, Z)
\end{tikzcd}
$$

It follows that (i) and (ii) are equivalent as stated.

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Next, suppose both $i : \partial D^m \rightarrowtail D^m$ and $j : \partial D^n \rightarrowtail D^n$ are generating cofibrations. Then we have a homeomorphism

$$\begin{tikzcd} \partial D^m \times D^n \amalg_{\partial D^m \times \partial D^n} D^m \times \partial D^n \arrow[r] \arrow[d, "\cong"] & D^m \times D^n \arrow[d, "\cong"] \\ \partial D^{m+n} \arrow[r, tail] & D^{m+n} \end{tikzcd}$$

which shows that the upper horizontal map is a cofibration as desired. In a similar manner, if $i : D^m \xrightarrow{\sim} D^m \times [0,1]$ is a trivial cofibration and $j : \partial D^n \rightarrowtail D^n$ a cofibration, the homeomorphism

$$\begin{tikzcd} D^m \times D^n \amalg_{D^m \times \partial D^n} D^m \times [0,1] \times \partial D^n \arrow[r] \arrow[d, "\cong"', "\Phi"] & D^m \times [0,1] \times D^n \arrow[d, "\cong"', "\Phi"] \\ D^{m+n} \arrow[r, tail, "\sim"] & D^{m+n} \times [0,1] \end{tikzcd}$$

shows that the upper horizontal map is a trivial cofibration. So (i) holds if both $i$ and $j$ are generating (trivial) cofibrations.

It follows that, if $p : Z \overset{\sim}{\twoheadrightarrow} W$ is a trivial fibration, a lifting exists in the diagram

$$
\begin{tikzcd}
\partial D^m \arrow[r] \arrow[d, tail] & \underline{\mathrm{Hom}}(D^n, Z) \arrow[d] \\
D^m \arrow[r] \arrow[ur, dashed] & \underline{\mathrm{Hom}}(D^n, W) \times_{\underline{\mathrm{Hom}}(\partial D^n, W)} \underline{\mathrm{Hom}}(\partial D^n, Z)
\end{tikzcd}
$$

But then a lifting exists in the diagram

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, tail, "i"] & \underline{\mathrm{Hom}}(D^n, Z) \arrow[d] \\
X \arrow[r] \arrow[ur, dashed] & \underline{\mathrm{Hom}}(D^n, W) \times_{\underline{\mathrm{Hom}}(\partial D^n, W)} \underline{\mathrm{Hom}}(\partial D^n, Z)
\end{tikzcd}
$$

Indeed, this follow immediately from the definition of Serre cofibrations. (Alternatively, the existence of a lifting in the upper diagram shows that the right-hand vertical map is a trivial fibration. But then a lift-

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ing exists in the lower diagram.) We conclude that a lifting exists in the diagram

$$\begin{tikzcd}
A \times D^n \underset{A \times \partial D^n}{\sqcup} X \times \partial D^n \arrow[r] \arrow[d] & Z \arrow[d, two heads, "p", "\sim"'] \\
X \times D^n \arrow[r] \arrow[ur, dashed] & W
\end{tikzcd},$$

and hence, in the diagram

$$\begin{tikzcd}
\partial D^n \arrow[r] \arrow[d, tail] & \underline{\mathrm{Hom}}(X, Z) \arrow[d] \\
D^n \arrow[r] \arrow[ur, dashed] & \underline{\mathrm{Hom}}(X, W) \underset{\underline{\mathrm{Hom}}(A, W)}{\times} \underline{\mathrm{Hom}}(A, Z)
\end{tikzcd}$$

But then a lifting exists in

$$\begin{tikzcd}
B \arrow[r] \arrow[d, tail] & \underline{\mathrm{Hom}}(X, Z) \arrow[d] \\
Y \arrow[r] & \underline{\mathrm{Hom}}(X, W) \underset{\underline{\mathrm{Hom}}(A, W)}{\times} \underline{\mathrm{Hom}}(A, Z)
\end{tikzcd},$$

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and hence, in

$$\begin{tikzcd} A\times Y \amalg_{A\times B} X\times B \arrow[r] \arrow[d] & Z \arrow[d, "\sim" ', "p"] \\ X\times Y \arrow[r] \arrow[ur, dashed] & W \end{tikzcd}$$

which shows that the left-hand vertical map is a cofibration as stated. A similar argument shows that this map is a trivial cofibration if either $i$ or $j$ is a trivial cofibration. This proves (i). //

We can now prove:

<u>Prop</u> Geometric realization

$$|-| : \mathcal{K}^{\Delta^{op}} \longrightarrow \mathcal{K}$$

is a left Quillen functor from the category of simplicial k-spaces with the Reedy model structure to the category of k-spaces with the Serre model structure.

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Proof We will show that the right adjoint functor

$$\underline{\mathrm{Sin}}(-)[-] : \mathcal{K} \longrightarrow \mathcal{K}^{\Delta^{op}}$$

takes (trivial) Serre fibrations to (trivial) Reedy fibrations. Recall that $p : X[-] \to Y[-]$ is a Reedy fibration if the induced map

$$X[n] \longrightarrow Y[n] \times_{M_n Y} M_n X$$

is a Serre fibration, for all $n \geq 0$. Now, we calculate

$$M_n \underline{\mathrm{Sin}}(Z) := \lim_{\substack{\eta : [m] \hookrightarrow [n] \\ m < n}} \underline{\mathrm{Sin}}(Z)[m]$$

$$= \lim_{\substack{\eta : [m] \hookrightarrow [n] \\ m < n}} \underline{\mathrm{Hom}}(\Delta[m], Z)$$

$$\xleftarrow{\cong} \underline{\mathrm{Hom}}\Big(\operatorname*{colim}_{\substack{\eta : [m] \hookrightarrow [n] \\ m < n}} \Delta[m], Z\Big)$$

$$\xleftarrow{\cong} \underline{\mathrm{Hom}}(\partial\Delta[m], Z).$$

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Hence, the map

$$\underline{\mathrm{Sin}}(Z)[n] \longrightarrow \underline{\mathrm{Sin}}(W)[n] \times_{M_n \underline{\mathrm{Sin}}(W)} M_n \underline{\mathrm{Sin}}(Z)$$

is homeomorphic to the map

$$\underline{\mathrm{Hom}}(\Delta[n], Z)$$

$$\longrightarrow \underline{\mathrm{Hom}}(\Delta[n], W) \times_{\underline{\mathrm{Hom}}(\partial\Delta[n], W)} \underline{\mathrm{Hom}}(\partial\Delta[n], Z)$$

By SM7, this map is a (trivial) fibration if $p: Z \to W$ is a (trivial) fibration. This proves the prop. //

<u>Cor</u> Let $f: X[-] \to Y[-]$ be a map between two Reedy cofibrant simplicial k-spaces and assume that, for all $n \geq 0$,

$$f: X[n] \longrightarrow Y[n]$$

is a weak equivalence. Then

$$|f|: |X[-]| \longrightarrow |Y[-]|$$

is a weak equivalence.

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_Proof_ Indeed, a left Quillen functor preserves weak equivalences between cofibrant objects; see p. 173. //

One can also use Reedy model structure arguments to prove the following result; see [Hovey, Lemma 5.2.6].

_Lemma_ (Gluing lemma) Let $\mathcal{C}$ be a model category, and let

$$\begin{tikzcd} X \arrow[d, "\sim"] & A \arrow[l] \arrow[r] \arrow[d, "\sim"] & B \arrow[d, "\sim"] \\ X' & A' \arrow[l] \arrow[r] & B' \end{tikzcd}$$

be a diagram of cofibrant objects such that the left-hand horizontal maps are cofibrations and the vertical maps weak equivalences. Then the induced map of push-outs

$$X \amalg_A B \longrightarrow X' \amalg_{A'} B'$$

is a weak equivalence. //

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