Let R be a unital associative ring. The Grothendieck group $K_0(R)$ is defined to be the abelian group generated by the finitely generated projective left R-modules subject to the relations that, for every short exact sequence

$$P' \rightarrowtail P \twoheadrightarrow P'',$$

the alternating sum

$$P' - P + P''$$

is equal to zero. This is a very important invariant of the ring R. We stress that only finitely generated projective R-modules are used in the definition. If, instead, we allowed all projective R-modules, the corresponding K-group would be zero. Indeed, for every projective R-module P, we have a short-exact sequence

$$0 \to P \to \bigoplus_{i \in \mathbb{N}} P \xrightarrow{s} \bigoplus_{i \in \mathbb{N}} P \to 0$$

where $\sigma(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$. So finiteness is essential.

Following Quillen and Waldhausen, we define a space $K(R)$ whose set of path-components $\pi_0(K(R), *)$ is canonically isomorphic to $K_0(R)$. We first introduce the following notion which is due to Waldhausen.

Def A category with cofibrations and weak equivalences is a category $\mathcal{C}$ together with a null object $*$ and two subcategories $co\,\mathcal{C}$ and $w\,\mathcal{C}$ such that:

cof 1: Every isomorphism is in $co\,\mathcal{C}$.

cof 2: For every $A \in ob\,\mathcal{C}$, $* \rightarrow A$ is in $co\,\mathcal{C}$.

cof 3: For every map $A \rightarrowtail B$ in $co\,\mathcal{C}$ and every map $A \to C$ in $\mathcal{C}$, the push-out $B \amalg_A C$ exists and the induced map $C \rightarrow B \amalg_A C$ is in $co\,\mathcal{C}$.

we 1: Every isomorphism is in $w\,\mathcal{C}$.

we 2: For every diagram

$$\begin{array}{ccccc} B & \leftarrowtail & A & \rightarrow & C \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ B' & \leftarrowtail & A' & \rightarrow & C' \end{array}$$

with the left-hand horizontal maps in co$\mathcal{E}$ and the vertical maps in w$\mathcal{E}$, the induced map

$$B \amalg_A C \longrightarrow B' \amalg_{A'} C'$$

is in w$\mathcal{E}$. //

The axioms cof 1 and we 1 imply, in particular, that

$$\mathrm{ob}(\mathrm{co}\,\mathcal{E}) = \mathrm{ob}(\mathrm{w}\,\mathcal{E}) = \mathrm{ob}\,\mathcal{E}.$$

<u>Ex</u> (i) The category P_R of finitely generated projective left modules over the ring R is a category with cofibrations and weak equivalences, where $f: P \to Q$ is a cofibration, if f is injective and $Q/f(P)$ is projective, and a weak

equivalence, if f is an isomorphism.

(ii) Let $\mathcal{M}$ be a pointed model category. Then the full subcategory $\mathcal{M}^c$ of cofibrant objects is a category with cofibrations and weak equivalences, where the cofibrations are the cofibrations and the weak equivalences are the weak equivalences. However, the K-theory of $\mathcal{M}^c$ will be trivial because no finiteness condition is imposed.

(iii) Let $\mathcal{M}$ be a pointed, cofibrantly generated model category, and let I be a set of generating cofibrations. Let $\mathcal{C}$ be the full subcategory of $\mathcal{M}$ whose objects are the retracts of finite I-cell complexes. Then $\mathcal{C}$ is a category with cofibrations and weak equivalences. It has an interesting K-theory. //

The K-theory of a category $\mathcal{C}$ with cofibrations and weak equivalences is defined by means of Waldhausen's S-construction which we now define.

Let $Ar[n]$ be the category of arrows in the category $[n]$. It is the category corresponding to the partially ordered set of all pairs (i,j) with $0 \leq i \leq j \leq n$ where $(i,j) \leq (i',j')$ if and only if $i \leq i'$ and $j \leq j'$. For example $Ar[2]$ is the partially ordered set

$$
\begin{array}{ccccc}
(0 \leq 0) & \leq & (0 \leq 1) & \leq & (0 \leq 2) \\
 & & \wedge\!| & & \wedge\!| \\
 & & (1 \leq 1) & \leq & (1 \leq 2) \\
 & & & & \wedge\!| \\
 & & & & (2 \leq 2)
\end{array}
$$

.

Def (Waldhausen) Let $\mathcal{E}$ be a category with cofibrations and weak equivalences, and let $n \geq 0$. We define a category $S\mathcal{E}[n]$ with cofibrations and weak equivalences as follows:

objects: the functors

$$A : Ar[n] \longrightarrow \mathcal{E}$$

$$(i,j) \longmapsto A_{i,j}$$

such that, for all $0 \le j \le n$,

$$A_{j,j} = * ,$$

and for all $0 \le i \le j \le k \le n$,

$$A_{i,j} \longrightarrow A_{i,k}$$

is in co $\mathcal{E}$ and

$$\begin{tikzcd} A_{i,j} \arrow[r, tail] \arrow[d] & A_{i,k} \arrow[d] \\ * = A_{j,j} \arrow[r, tail] & A_{j,k} \end{tikzcd}$$

is a push-out in $\mathcal{E}$.

morphisms : natural transformations

cofibrations : morphisms

$$A \longrightarrow A'$$

such that, for all $0 \le i \le j \le k \le n$,

$$A_{i,j} \longrightarrow A'_{i,j}$$

and

$$A_{i,k} \amalg_{A_{i,j}} A'_{i,j} \longrightarrow A'_{i,k}$$

are in $co\,\mathcal{E}$.

weak equivalences: morphisms

$$A \longrightarrow A'$$

such that, for all $0 \le i \le j \le n$,

$$A_{i,j} \longrightarrow A'_{i,j}$$

is in $w\mathcal{E}$. //

It is not completely trivial to prove that (a) $S\mathcal{E}[n]$ is a category and (b) $S\mathcal{E}[n]$ is a category with cofibrations and weak equivalences. We refer to §1.1–1.2 of Waldhausen's paper "Algebraic K-theory of spaces" in LNM 1126 for the proof. (The paper is available online from Waldhausen's homepage.)

A non-decreasing map

$$\theta : [m] \longrightarrow [n]$$

induces a functor

$$\theta_* : Ar[m] \longrightarrow Ar[n]$$

which, in turn, induces a functor

$$\theta^* : S\mathcal{E}[n] \longrightarrow S\mathcal{E}[m].$$

The functor θ^* is <u>exact</u>: it takes $*$ to $*$, cofibrations to cofibrations, weak equivalences to weak equivalences, and push-outs along a cofibration to push-outs. Hence

$$[n] \longmapsto S\mathcal{E}[n]$$

is a simplicial object in the category of categories with cofibrations and weak equivalences and exact functors. Hence, we may iterate the construction:

$$S^{(n)}\mathcal{E}[-] := \underbrace{S \ldots S}_{n}\mathcal{E}[-].$$

Ex $S\mathcal{E}[0]$: one object

$$A_{0,0} = *$$

and one morphism.

$S\mathcal{E}[1]$: objects :

$$\begin{tikzcd} * = A_{0,0} \arrow[r, tail] & A_{0,1} \arrow[d] \\ & * = A_{1,1} \end{tikzcd}$$

morphisms : maps of diagrams

$\therefore$ $S\mathcal{E}[1] \xrightarrow{\sim} \mathcal{E}$; $A \longmapsto A_{0,1}$.

$S\mathcal{E}[2]$: objects :

$$\begin{tikzcd} * = A_{0,0} \arrow[r, tail] & A_{0,1} \arrow[r, tail] \arrow[d] & A_{0,2} \arrow[d] \\ & * = A_{1,1} \arrow[r, tail] & A_{1,2} \arrow[d] \\ & & * = A_{2,2} \end{tikzcd}$$

morphisms : maps of diagrams.

$d^0 : [1] \to [2] : 0 \mapsto 1,\ 1 \mapsto 2.$

$d^0_* : Ar[1] \to Ar[2]$

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[drrrr, mapsto, bend left] & (0,1) \arrow[d] \arrow[drrrr, mapsto] & & (0,0) \arrow[r] & (0,1) \arrow[r] \arrow[d] & (0,2) \arrow[d] \\
& (1,1) \arrow[drrrr, mapsto] & & & (1,1) \arrow[r] & (1,2) \arrow[d] \\
& & & & & (2,2)
\end{tikzcd}$$

$d_0 = (d^0)^* : S\mathcal{E}[2] \to S\mathcal{E}[1]$

$$\begin{tikzcd}
* = A_{0,0} \arrow[r, tail] & A_{0,1} \arrow[r, tail] \arrow[d] & A_{0,2} \arrow[d] & & & \\
& * = A_{1,1} \arrow[r, tail] & A_{1,2} \arrow[d] \arrow[r, mapsto] & * = A_{1,1} \arrow[r, tail] & A_{1,2} \arrow[d] \\
& & * = A_{2,2} & & * = A_{2,2}
\end{tikzcd}$$

$d^1 : [1] \to [2] : 0 \mapsto 0,\ 1 \mapsto 2.$

$$\begin{tikzcd}
* = A_{0,0} \arrow[r, tail] & A_{0,1} \arrow[r, tail] \arrow[d] & A_{0,2} \arrow[d] & & & \\
& * = A_{1,1} \arrow[r, tail] & A_{1,2} \arrow[d] \arrow[r, mapsto] & * = A_{0,0} \arrow[r, tail] & A_{0,2} \arrow[d] \\
& & * = A_{2,2} & & * = A_{2,2}
\end{tikzcd}$$

$d^2 : [1] \to [2] \ ; \ 0 \mapsto 0, \ 1 \mapsto 1$

$$\begin{tikzcd}[column sep=small]
* = A_{0,0} \arrow[r, tail] & A_{0,1} \arrow[r, tail] \arrow[d] & A_{0,2} \arrow[d] & & * = A_{0,0} \arrow[r, tail] & A_{0,1} \arrow[d] \\
 & * = A_{1,1} \arrow[r, tail] & A_{1,2} \arrow[d] & \longmapsto & & * = A_{1,1} \\
 & & * = A_{2,2} & & &
\end{tikzcd}$$

$s^0 : [2] \to [1] \ , \ 0, 1 \mapsto 0, \ 2 \mapsto 1$

$$\begin{tikzcd}[column sep=small]
* = A_{0,0} \arrow[r, tail] & A_{0,1} \arrow[d] & & * = A_{0,0} \arrow[r, equal] & A_{0,0} \arrow[r, tail] \arrow[d, equal] & A_{0,1} \arrow[d, equal] \\
 & * = A_{1,1} & \longmapsto & & * = A_{0,0} \arrow[r, tail] & A_{0,1} \arrow[d] \\
 & & & & & * = A_{1,1}
\end{tikzcd}$$

$s^1 : [2] \to [1] \ , \ 0 \mapsto 0 ; \ 1, 2 \mapsto 1 .$

$$\begin{tikzcd}[column sep=small]
* = A_{0,0} \arrow[r, tail] & A_{0,1} \arrow[d] & & * = A_{0,0} \arrow[r, tail] & A_{0,1} \arrow[r, equal] \arrow[d] & A_{0,1} \arrow[d] \\
 & * = A_{1,1} & \longmapsto & & * = A_{1,1} \arrow[r, equal] & A_{1,1} \arrow[d, equal] \\
 & & & & & * = A_{1,1}
\end{tikzcd}$$

//

We define

$$K(\mathcal{E})_n = |N_w S^{(n)} \mathcal{E}[-]|$$

This is the geometric realization of an $(n+1)$-simplicial set. We can realize the $n+1$ simplicial directions one at a time or we can realize the diagonal simplicial set. The resulting spaces are canonically homeomorphic by the theorem on p. 202. Let us now view $N_w S^{(n)} \mathcal{E}[-]$, where $n \geq 1$, as a bi-simplicial set

$$([k],[\ell]) \longmapsto N_w S^{(n)} \mathcal{E}[k,\ell]$$

where ℓ represents the first S-direction and k represents the diagonal of the remaining simplicial directions. Then

$$K(\mathcal{E})_n = |[k] \mapsto |[\ell] \mapsto N_w S^{(n)} \mathcal{E}[k,\ell]||$$

Now, since $S\mathcal{E}[0] = *$ and $S\mathcal{E}[1] = \mathcal{E}$, we have

$$\mathrm{sk}_1 \, | \, [\ell] \mapsto N_w S^{(n)} \mathcal{E}[k,\ell] \, |$$

$$\cong \frac{N_w S^{(n-1)} \mathcal{E}[k] \times \Delta[1]}{N_w S^{(n-1)} \mathcal{E}[k] \times \partial\Delta[1] \cup * \times \Delta[1]}$$

$$\cong N_w S^{(n-1)} \mathcal{E}[k] \wedge S^1$$

and hence

$$| \, [k] \mapsto \mathrm{sk}_1 \, | \, [\ell] \mapsto N_w S^{(n)} \mathcal{E}[k,\ell] \, | \, |$$

$$\cong | \, [k] \mapsto N_w S^{(n-1)} \mathcal{E}[k] \wedge S^1 \, |$$

$$\cong K(\mathcal{E})_{n-1} \wedge S^1 .$$

It follows that the inclusion of the 1-skeleton in the ℓ-direction gives rise to a canonical map

$$\sigma : \Sigma K(\mathcal{E})_{n-1} \longrightarrow K(\mathcal{E})_n$$

for all $n \geq 1$. By adjunction, the maps σ are equivalent to maps

$$\sigma^\flat : K(\mathcal{E})_{n-1} \longrightarrow \Omega K(\mathcal{E})_n .$$

In general, a spectrum E is a sequence of pointed spaces E_n, $n \geq 0$, together with maps of pointed spaces

$$\sigma : \Sigma E_{n-1} \longrightarrow E_n$$

called the structure maps. The homotopy groups of the spectrum E are defined by

$$\pi_q(E) := \operatorname*{colim}_n \pi_{n+q}(E_n, *).$$

We remark that the definition of $\pi_q(E)$ makes sense, for all integers q, and that $\pi_q(E)$ is an abelian group, for all integers q. (Spectra play a role in homotopy theory analogous to abelian groups in algebra, but this is not easy to see from the definition.)

Def Let $\mathcal{E}$ be a category with cofibrations and weak equivalences. The algebraic K-theory of $\mathcal{E}$ is the spectrum

$$K(\mathcal{E}) = \{ K(\mathcal{E})_n \}$$

and the algebraic K-groups of $\mathcal{E}$ are the homotopy groups

$$K_q(\mathcal{E}) := \pi_q(K(\mathcal{E})). \qquad //$$

If R is a ring, we define

$$K(R) := K(\mathcal{P}_R)$$

to be the K-theory of the category with cofibrations and weak equivalences $\mathcal{P}_R$ given by the finitely generated projective left R-modules. We define a map

$$K_0(R) \longrightarrow \pi_0(K(R))$$

from the Grothendieck group. Since $K(\mathcal{E})_0 = |Nw\mathcal{E}[-1]|$ and $Nw\mathcal{E}[0] = \mathrm{ob}\,\mathcal{E}$, we have a canonical map

$$\mathrm{ob}\,\mathcal{E} = sk_0 K(\mathcal{E})_0 \to K(\mathcal{E})_0$$

$$\to \pi_0(K(\mathcal{E})_0, *) \longrightarrow \pi_0(K(\mathcal{E})).$$

We claim that, for $\mathcal{E} = \mathcal{P}_R$, it factors through the relation that, for every short-exact sequence,

$$P_{0,1} \rightarrowtail P_{0,2} \twoheadrightarrow P_{1,2} .$$

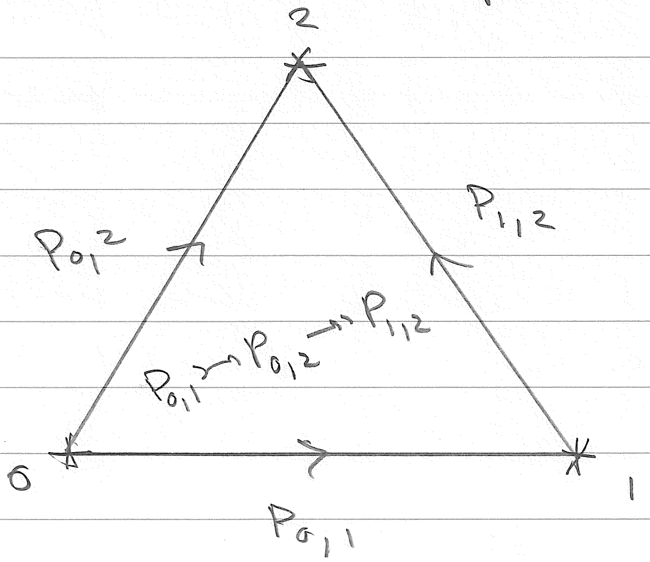
the alternating sum

$$P_{0,1} - P_{0,2} + P_{1,2}$$

is equal to zero. Indeed, the short-exact sequence defines an element of $S\mathcal{E}[2]$. Therefore, in the space

$$|\,[\ell] \mapsto N w S\mathcal{E}[0,\ell]\,|$$

we have a 2-simplex



and hence, a homotopy from the loop $\sigma(P_{0,2})$ to the loop $\sigma(P_{1,2}) * \sigma(P_{0,1})$.

It follows that

$$\mathrm{ob}\, \mathcal{P}_R \to \pi_0(K(\mathcal{C}_0), *) \to \pi_1(K(\mathcal{C})_1, *)$$

factors through $K_0(R)$. We will prove later that the map

$$K_0(R) \to \pi_0(K(R))$$

is an isomorphism. We will also show the following general result.

Thm Let $\mathcal{C}$ be a category with cofibrations and weak equivalences. Then the structure maps

$$\sigma^{\#} : K(\mathcal{C})_n \to \Omega K(\mathcal{C})_{n+1}$$

are weak equivalences, for $n \geq 1$. /