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Let $\mathcal{E}$ be a category with cofibrations and weak equivalences. The three exact functors

$$d_0, d_1, d_2 : S\mathcal{E}[2] \longrightarrow S\mathcal{E}[1] = \mathcal{E}$$

associate to the cofibration sequence

$$A_{0,1} \rightarrowtail A_{0,2} \twoheadrightarrow A_{1,2}$$

the quotient object $A_{1,2}$, the total object $A_{0,2}$, and the subobject $A_{0,1}$, respectively. We will prove:

<u>Thm</u> (Additivity theorem) Let $\mathcal{E}$ be a category with cofibrations and weak equivalences. Then the exact functor

$$(d_2, d_0) : S\mathcal{E}[2] \longrightarrow \mathcal{E} \times \mathcal{E}$$

induce a weak equivalence

$$K(S\mathcal{E}[2])_n \xrightarrow{\sim} K(\mathcal{E} \times \mathcal{E})_n,$$

for all $n \geq 1$. /

Since $S^{(n-1)}\mathcal{E}$ is a simplicial catewith cofibrations and weak equivalences, the lemma on p. 210 shows that it suffices to prove the theorem for $n=1$. This takes some preparation.

We consider the following functors and natural transformations:

$$\begin{tikzcd}
\Delta\times\Delta \arrow[rr, "\amalg"] \arrow[rr, bend left=60, "\mathrm{pr}_1"] \arrow[rr, bend right=60, "\mathrm{pr}_2"'] \arrow[rr, phantom, bend left=30, "\varepsilon_L\Downarrow\ \Uparrow\iota_L"] \arrow[rr, phantom, bend right=30, "\varepsilon_R\Uparrow\ \Downarrow\iota_R"] & & \Delta
\end{tikzcd}$$

The functor $\amalg$ is called concatenation and is defined by

$$\begin{tikzcd}
{[m]\amalg[n]} \arrow[d, "\theta\amalg\eta"] & := & {[m+n+1]} \arrow[d] \\
{[m']\amalg[n']} & := & {[m'+n'+1]}
\end{tikzcd}$$

$$(\theta\amalg\eta)(i)=\begin{cases}\theta(i) & (0\leq i\leq m)\\ \eta(i-m-1)+m'+1 & (m+1\leq i\leq m+n+1)\end{cases}$$

The natural transformations

$$\begin{array}{ccccc} pr_1([m],[n]) & \underset{\iota_L}{\overset{\varepsilon_L}{\rightleftarrows}} & [m] \amalg [n] & \underset{\iota_R}{\overset{\varepsilon_R}{\leftrightarrows}} & pr_2([m],[n]) \\ \| & & \| & & \| \\ [m] & \rightleftarrows & [m+n+1] & \leftrightarrows & [n] \end{array}$$

are defined by

$$\varepsilon_L(i) = i \quad ; \quad \varepsilon_R(i) = m+1+i$$

$$\iota_L(i) = \begin{cases} i & (0 \leq i \leq m) \\ m & (m+1 \leq i \leq m+n+1) \end{cases}$$

$$\iota_R(i) = \begin{cases} 0 & (0 \leq i \leq m) \\ i-(m+1) & (m+1 \leq i \leq m+n+1) \end{cases}.$$

Let $X[-]$ be a simplicial set. We then define the bi-simplicial sets

$$\begin{array}{ccccc} XL[-,-] & \xleftarrow{\varepsilon_L^*} & DX[-,-] & \xrightarrow{\varepsilon_R^*} & XR[-,-] \\ \| & & \| & & \| \\ X[-]\circ pr_1 & \xleftarrow{\varepsilon_L^*} & X[-]\circ \amalg & \xrightarrow{\varepsilon_R^*} & X[-]\circ pr_2 \end{array}$$

Let $f, g : X[-] \to Y[-]$ be two maps of simplicial sets. A <u>simplicial homotopy</u> from f to g is a commutative diagram of maps of simplicial sets

$$\begin{tikzcd}
X[-]\times\Delta[0][-] \arrow[r, equal] \arrow[d, "id\times d^0"] & X[-] \arrow[d, "f"] \\
X[-]\times\Delta[1][-] \arrow[r, "h"] & Y[-] \\
X[-]\times\Delta[0][-] \arrow[r, equal] \arrow[u, "id\times d^1"'] & X[-] \arrow[u, "g"']
\end{tikzcd}$$

<u>Lemma</u> For all $m \geq 0$ and $n \geq 0$, there exist natural simplicial homotopies from the composite maps

$$DX[m,-] \xrightarrow{\varepsilon_L^*} XL[m,-] \xrightarrow{\iota_L^*} DX[m,-]$$

$$DX[-,n] \xrightarrow{\varepsilon_R^*} XR[-,n] \xrightarrow{\iota_R^*} DX[-,n]$$

to the respective identity maps.

<u>Proof</u> We prove the statement for the second map. It is induced by the map $\varepsilon_R \circ \iota_R$. We define

$$\begin{array}{ccc} h : DX[m,n] \times \Delta[1][m] & \longrightarrow & DX[m,n] \\ \| & & \| \\ X[m+n+1] \times \mathrm{Hom}_{\Delta}([m],[1]) & \longrightarrow & X[m+n+1] \end{array}$$

to be the adjoint of the map

$$\mathrm{Hom}_{\Delta}([m],[1]) \xrightarrow{\tilde{h}} \mathrm{Map}(X[m+n+1], X[m+n+1])$$

given by $\tilde{h}(a) = \varphi_m(a)^*$ where

$$\varphi_m(a) \in \mathrm{Hom}_{\Delta}(X[m+n+1], X[m+n+1])$$

is the map that, for $0 \le i \le m$, is defined by

$$\varphi_m(a)(i) = \begin{cases} i & (a(i) = 0) \\ m+1 & (a(i) = 1) \end{cases}$$

and, for $m+1 \le i \le m+n+1$, by

$$\varphi_m(a)(i) = i.$$

To prove that h is a simplicial map, we must show that, for every map $\theta : [m'] \to [m]$ in Δ, the following diagram commutes:

$$\begin{tikzcd}
X[m+n+1] \times \Delta[1][m] \arrow[r, "h"] \arrow[d, "(\theta \amalg id)^* \times \theta^*"] & X[m+n+1] \arrow[d, "(\theta \amalg id)^*"] \\
X[m'+n+1] \times \Delta[1][m'] \arrow[r, "h"] & X[m'+n+1]
\end{tikzcd}.$$

By adjunction, this is equivalent to showing that that

$$\begin{tikzcd}
\Delta[1][m] \arrow[r, "\tilde{h}"] \arrow[dd, "\theta^*"] & \mathrm{Map}(X[m+n+1], X[m+n+1]) \arrow[d, "\mathrm{Map}(id, (\theta \amalg id)^*)"] \\
& \mathrm{Map}(X[m+n+1], X[m'+n+1]) \\
\Delta[1][m'] \arrow[r, "\tilde{h}"] & \mathrm{Map}(X[m'+n+1], X[m'+n+1]) \arrow[u, "\mathrm{Map}((\theta \amalg id)^*, id)"']
\end{tikzcd}$$

commutes, By the definition of $\tilde{h}$, this is equivalent to showing that

$$\begin{tikzcd}
\Delta[1][m] \arrow[r, "\varphi_m"] \arrow[dd, "\theta^*"] & \mathrm{Hom}_{/\Delta}([m] \amalg [n], [m] \amalg [n]) \arrow[d, "\mathrm{Hom}(\theta \amalg id, id)"] \\
& \mathrm{Hom}_{/\Delta}([m'] \amalg [n], [m] \amalg [n]) \\
\Delta[1][m'] \arrow[r, "\varphi_{m'}"] & \mathrm{Hom}_{/\Delta}([m'] \amalg [n], [m'] \amalg [n]) \arrow[u, "\mathrm{Hom}(id, \theta \amalg id)"']
\end{tikzcd}$$

commutes. This, in turn, is equivalent to showing that, for all $a : [m] \to [1]$, the following diagram commutes

$$\begin{array}{ccc} [m'] \amalg [n] & \xrightarrow{\varphi_{m'}(a \circ \theta)} & [m'] \amalg [n] \\ \downarrow \theta \amalg \mathrm{id} & & \downarrow \theta \amalg \mathrm{id} \\ [m] \amalg [n] & \xrightarrow{\varphi_m(a)} & [m] \amalg [n] . \end{array}$$

We leave it as an exercise to check that this is true. Finally, the diagram

$$\begin{array}{ccc} DX[-,n] \times \Delta[0][-] & = & DX[-,n] \\ \downarrow \mathrm{id} \times d^0 & & \downarrow \iota_R^* \circ \varepsilon_R^* \\ DX[-,n] \times \Delta[1][-] & \xrightarrow{h} & DX[-,n] \\ \uparrow \mathrm{id} \times d^1 & & \uparrow \mathrm{id} \\ DX[-,n] \times \Delta[0][-] & = & DX[-,n] \end{array}$$

commutes. //

The geometric realization of a simplicial homotopy is a homotopy. Moreover, $\varepsilon_L^* \circ \iota_L^* = \mathrm{id}$ and $\varepsilon_R^* \circ \iota_R^* = \mathrm{id}$. Hence, the two maps

$$|DX[m,-]| \xrightarrow{\varepsilon_L^*} |XL[m,-]| = X[m]$$

$$|DX[-,n]| \xrightarrow{\varepsilon_R^*} |XR[-,n]| = X[n]$$

are weak equivalences, for all m and n, respectively. As m varies, the maps ε_L^* form a map of simplicial spaces. Similarly, the maps ε_R^* constitute a map of simplicial spaces. (This is not true for the maps ι_L^* and ι_R^*.) By the lemma on p. 210, we conclude that the induced maps

$$|X[-]| \xleftarrow{\varepsilon_L^*} |DX[-,-]| \xrightarrow{\varepsilon_R^*} |X[-]|$$

are both weak equivalences.

Now, let $f: \mathcal{C} \to \mathcal{D}$ be an exact functor between two categories with cofibrations and weak equivalences. We consider the following diagram of bi-simplicial categories with cofibrations and weak equivalences, where the upper left-hand term is defined to be the pull-back:

$$\begin{tikzcd} S f | \mathcal{D}[-,-] \arrow[r, "f'"] \arrow[d, "\varepsilon_L'"] & DS\mathcal{D}[-,-] \arrow[d, "\varepsilon_L^*"] \\ (S\mathcal{C})L[-,-] \arrow[r, "f"] & (S\mathcal{D})L[-,-] \end{tikzcd}$$

If we ignore choices of quotient objects, an object of $Sf|\mathcal{D}[m,n]$ is a pair of flags in $\mathcal{C}$ and $\mathcal{D}$

$$C_1 \rightarrowtail \cdots \rightarrowtail C_m$$

$$D_1 \rightarrowtail \cdots \rightarrowtail D_m \rightarrowtail S_0 \rightarrowtail \cdots \rightarrowtail S_n$$

such that $f(C_i) = D_i$, $1 \le i \le m$. The map ε_L' forgets the second row. For fixed m, the section

$$\iota_L^* : (S\mathcal{D})[m,-] \longrightarrow DS\mathcal{D}[-,-]$$

induces a section

$$\iota_L' : (S\mathcal{C})[m,-] \longrightarrow Sf|\mathcal{D}[m,-]$$

which maps

$$C_1 \rightarrowtail \cdots \rightarrowtail C_m$$

to

$$C_1 \rightarrowtail \cdots \rightarrowtail C_m$$

$$D_1 \rightarrowtail \cdots \rightarrowtail D_m = D_m = \cdots = D_m$$

where $D_i = f(C_i)$, $1 \le i \le m$. The homotopy from the lemma above induces a simplicial homotopy from

$$Sf|\mathcal{D}[m,-] \xrightarrow{\varepsilon_L'} (S\mathcal{E})L[m,-] \xrightarrow{\iota_L'} Sf|\mathcal{D}[m,-]$$

to the identity. Moreover, for fixed n, the are simplicial exact functors

$$\begin{array}{ccccc}
DS\mathcal{D}[-,n] & \xrightarrow{\varepsilon_L^*} & (S\mathcal{D})L[-,n] & \xleftarrow{f} & (S\mathcal{C})L[-,n] \\
\downarrow \varepsilon_R^* & & \downarrow & & \downarrow \\
S\mathcal{D}[n] & \longrightarrow & * & \longleftarrow & * \\
\downarrow \iota_R^* & & \downarrow & & \downarrow \\
DS\mathcal{D}[-,n] & \xrightarrow{\varepsilon_L^*} & (S\mathcal{D})L[-,n] & \xleftarrow{f} & (S\mathcal{C})L[-,n]
\end{array}$$

and, taking horizontal pull-backs, we get simplicial exact functors

$$S\mathcal{f}|\mathcal{D}[-,n] \xrightarrow{\varepsilon_R'} S\mathcal{D}[n] \xrightarrow{\iota_L'} S\mathcal{f}|\mathcal{D}[-,n]$$

that takes

$$C_1 \rightarrowtail \cdots \rightarrowtail C_m$$

$$D_1 \rightarrowtail \cdots \rightarrowtail D_m \rightarrowtail S_0 \rightarrowtail \cdots \rightarrowtail S_n$$

to

$$* = \cdots = *$$

$$* = \cdots = * = S_0/S_0 \rightarrowtail \cdots \rightarrowtail S_n/S_0.$$

We now have the following result similar to Quillen's theorem A:

<u>Prop</u> Suppose that, for all $n \geq 0$, the

$$|\mathrm{ob}\, S\mathcal{f}|\mathcal{D}[-,n]| \xrightarrow{\iota_R' \circ \varepsilon_R'} |\mathrm{ob}\, S\mathcal{f}|\mathcal{D}[-,n]|$$

is a weak equivalence. Then

$$|\mathrm{ob}\, S\mathcal{E}[-]| \xrightarrow{f} |\mathrm{ob}\, S\mathcal{D}[-]|$$

is a weak equivalence.

Proof This follows from the following diagram:

$$\begin{tikzcd}
{|obS\mathcal{E}[-1]|} \arrow[d, "f"] & {|obSf|\mathcal{D}[-,-1]|} \arrow[l, "\varepsilon_L'"', "\sim"] \arrow[r, "\varepsilon_R'"] \arrow[d, "f'"] & {|obS\mathcal{D}[-1]|} \arrow[d, equal] \\
{|obS\mathcal{D}[-1]|} & {|obDS\mathcal{D}[-,-1]|} \arrow[l, "\varepsilon_L^*"', "\sim"] \arrow[r, "\varepsilon_R^*", "\sim"'] & {|obS\mathcal{D}[-1]|}
\end{tikzcd}$$

The maps marked by $\sim$ are weak equivalences by the lemmas on pp. 238 and 210. Finally, the assumption and the lemma on p. 210 show that the map ε_R', too, is a weak equivalence. Hence, the left-hand vertical map f is a weak equivalence as stated. //

Proof (of the additivity theorem) We wish to show that

$$\begin{array}{ccc}
K(S\mathcal{E}[2])_1 & \xrightarrow{(d_2,\, d_0)} & K(\mathcal{E}\times\mathcal{E})_1 \\
\| & & \| \\
|NwS(S\mathcal{E}[2])[-,-1]| & \longrightarrow & |NwS(\mathcal{E}\times\mathcal{E})[-,-1]|
\end{array}$$

is a weak equivalence. We first

show that it suffices to show that, for all $\mathcal{E}$, the map

$$|\mathrm{ob}\, S(S\mathcal{E}[2])[-1]| \xrightarrow{(d_2, d_0)} |\mathrm{ob}\, S(\mathcal{E}\times\mathcal{E})[-1]|$$

is a weak equivalence. We define $\mathbb{N}\mathcal{E}[n]$ to be the functor category

$$\mathbb{N}\mathcal{E}[n] = \mathcal{E}^{[n]}.$$

So $\mathrm{ob}(\mathbb{N}\mathcal{E}[n]) = N\mathcal{E}[n]$. We also define the category

$$\mathbb{N}^w\mathcal{E}[n] \subset \mathbb{N}\mathcal{E}[n]$$

to be the <u>full</u> subcategory with

$$\mathrm{ob}(\mathbb{N}^w\mathcal{E}[n]) = Nw\mathcal{E}[n].$$

A map in $\mathbb{N}^w\mathcal{E}[n]$ is a diagram

$$\begin{array}{ccccccc}
C_0 & \xrightarrow{\sim} & C_1 & \xrightarrow{\sim} & \cdots & \xrightarrow{\sim} & C_n \\
\downarrow & & \downarrow & & & & \downarrow \\
C_0' & \xrightarrow{\sim} & C_1' & \xrightarrow{\sim} & \cdots & \xrightarrow{\sim} & C_n'
\end{array}$$

where the horizontal maps are weak

equivalences but where the vertical maps are allowed to be any maps that make the diagram commute. We define this map to be a cofibration (resp. weak equivalence) if all the vertical maps are cofibrations (resp. weak equivalences) in the category $\mathcal{E}$. This makes $\mathbb{N}^w \mathcal{E}[-]$ a simplicial category with cofibrations and weak equivalences. Moreover, we have a canonical isomorphism of simplicial categories with cofibrations and weak equivalences

$$\mathbb{N}^w S\mathcal{E}[-] \cong S\mathbb{N}^w \mathcal{E}[-].$$

Hence,

$$N_w S\mathcal{E}[-] = \mathrm{ob}(\mathbb{N}^w S\mathcal{E}[-])$$

$$\cong \mathrm{ob}\, S\mathbb{N}^w \mathcal{C}[-].$$

Therefore, by the lemma on p. 210, to show that

$$|N_w S(S\mathcal{E}[2])[-]| \xrightarrow{(d_2, d_0)} |N_w S(\mathcal{E}\times\mathcal{E})|$$

is a weak equivalence, it suffices to show that, for all $n \geq 0$,

$$|\operatorname{ob} S(S(\mathbb{N}^w \mathcal{C}[n])[2])[-]| \xrightarrow{(d_2, d_0)} |\operatorname{ob} S(\mathbb{N}^w \mathcal{C}[n] \times \mathbb{N}^w \mathcal{C}[n])[-]|$$

is a weak equivalence. So it is enough to show that, for all $\mathcal{E}$,

$$|\operatorname{ob} S(S\mathcal{E}[2])[-]| \xrightarrow{(d_2, d_0)} |\operatorname{ob} S(\mathcal{E} \times \mathcal{E})[-]|$$

is a weak equivalence as stated.

To prove this, we apply the prop. on p. 245 to the functor

$$S\mathcal{E}[2] \xrightarrow{(d_2, d_0)} \mathcal{E} \times \mathcal{E}$$

$$(A \rightarrowtail C \twoheadrightarrow B) \longmapsto (A, B)$$

We must show that, for fixed $n \geq 0$, the map that to the object

$$\begin{array}{ccccccc}
A_1 & \rightarrowtail & A_2 & \rightarrowtail & \cdots & \rightarrowtail & A_m \\
\tilde{\downarrow} & & \tilde{\downarrow} & & & & \tilde{\downarrow} \\
C_1 & \rightarrowtail & C_2 & \rightarrowtail & \cdots & \rightarrowtail & C_m \\
\downarrow & & \downarrow & & & & \downarrow \\
B_1 & \rightarrowtail & B_2 & \rightarrowtail & \cdots & \rightarrowtail & B_m
\end{array}$$

$$A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_m \rightarrowtail S_0 \rightarrowtail S_1 \rightarrowtail \cdots \rightarrowtail S_n$$

$$B_1 \rightarrowtail B_2 \rightarrowtail \cdots \rightarrowtail B_m \rightarrowtail T_0 \rightarrowtail T_1 \rightarrowtail \cdots \rightarrowtail T_n$$

assigns

$$\begin{array}{ccccccc}
* & = & * & = & \cdots & = & * \\
\tilde{\downarrow} & & \tilde{\downarrow} & & & & \tilde{\downarrow} \\
* & = & * & = & \cdots & = & * \\
\downarrow & & \downarrow & & & & \downarrow \\
* & = & * & = & \cdots & = & *
\end{array}$$

$$* = * = \cdots = * = S_0/S_0 \rightarrowtail S_1/S_0 \rightarrowtail \cdots \rightarrowtail S_n/S_0$$

$$* = * = \cdots = * = T_0/T_0 \rightarrowtail T_1/T_0 \rightarrowtail \cdots \rightarrowtail T_n/T_0$$

induces a weak equivalence

$$|\operatorname{ob} S(d_2, d_0)|(\mathcal{E} \times \mathcal{E})[-, n]|$$

$$\xrightarrow{\sim} |\operatorname{ob} S(d_2, d_0)|(\mathcal{E} \times \mathcal{E})[-, n]|.$$

We first give a simplicial homotopy from the map that to the object on the top of p. 250 assigns the object

$$\begin{tikzcd}
* \arrow[r, equal] \arrow[d, tail] & * \arrow[r, equal] \arrow[d, tail] & \cdots \arrow[r, equal] & * \arrow[d, tail] \\
B_1 \arrow[r, tail] \arrow[d, equal] & B_2 \arrow[r, tail] \arrow[d, equal] & \cdots \arrow[r, tail] & B_m \arrow[d, equal] \\
B_1 \arrow[r, tail] & B_2 \arrow[r, tail] & \cdots \arrow[r, tail] & B_m
\end{tikzcd}$$

$$\begin{tikzcd}
* \arrow[r, equal] & * \arrow[r, equal] & \cdots \arrow[r, equal] & * \arrow[r, equal] & S_0/S_0 \arrow[r, tail] & S_1/S_0 \arrow[r, tail] & \cdots \arrow[r, tail] & S_n/S_0 \\
B_1 \arrow[r, tail] & B_2 \arrow[r, tail] & \cdots \arrow[r, tail] & B_m \arrow[r, tail] & T_0 \arrow[r, tail] & T_1 \arrow[r, tail] & \cdots \arrow[r, tail] & T_n
\end{tikzcd}$$

Let $X_i = C_i \amalg_{A_i} S_0$ such that

$$\begin{tikzcd}
A_i \arrow[r, tail] \arrow[d, tail] & S_0 \arrow[d, tail] \\
C_i \arrow[r, tail] \arrow[d, two heads] & X_i \arrow[d, equal] \\
B_i \arrow[r, equal] & B_i
\end{tikzcd}$$

Then for $a = 1 : [m] \to [1]$, we use the map above. If $a(0) = \cdots = a(i) = 0$ and $a(i+1) = \cdots = a(m) = 1$, $0 \leq i < m$, we map the object on top of p. 250 to

$$\begin{array}{ccccccccccc}
A_1 & \rightarrowtail & \cdots & \rightarrowtail & A_i & \rightarrowtail & S_0 & = & \cdots & = & S_0 \\
\downarrow & & & & \downarrow & & \downarrow & & & & \downarrow \\
C_1 & \rightarrowtail & \cdots & \rightarrowtail & C_i & \rightarrowtail & X_{i+1} & \rightarrowtail & \cdots & \rightarrowtail & X_m \\
\downarrow & & & & \downarrow & & \downarrow & & & & \downarrow \\
B_1 & \rightarrowtail & \cdots & \rightarrowtail & B_i & \rightarrowtail & B_{i+1} & \rightarrowtail & \cdots & \rightarrowtail & B_m
\end{array}$$

$$A_1 \rightarrowtail \cdots \rightarrowtail A_i \rightarrowtail S_0 = \cdots = S_0 = S_0 \rightarrowtail \cdots \rightarrowtail S_n$$

$$B_1 \rightarrowtail \cdots \rightarrowtail B_i \rightarrowtail B_{i+1} \rightarrowtail \cdots \rightarrowtail B_m \rightarrowtail T_0 \rightarrowtail \cdots \rightarrowtail T_n .$$

This defines a simplicial homotopy

$$S(d_2, d_0) \mid (\mathcal{E} \times \mathcal{E})[-] \times \Delta[1][-]$$

$$\longrightarrow S(d_2, d_0) \mid (\mathcal{E} \times \mathcal{E})[-]$$

from the map that takes the object on top p. 250 to the object on top. p. 251 to the identity map. We next define a simplicial homotopy from the map on page 250 to the map above. For $a = 1 : [m] \to [1]$ it is the map on p. 250. For $a : [m] \to [1]$ with $a(0) = \cdots = a(i) = 0$ and $a(i+1) = \cdots = a(m) = 1$, $0 \le i < m$, it takes the object on top p. 250 to the object

$$\begin{tikzcd}
* \arrow[r, equal] & \cdots \arrow[r, equal] & * \arrow[r, equal] \arrow[d, tail] & * \arrow[r, equal] \arrow[d, tail] & \cdots \arrow[r, equal] & * \arrow[d, tail] \\
B_1 \arrow[r, tail] \arrow[u, phantom] & \cdots \arrow[r, tail] & B_i \arrow[r, tail] \arrow[d, equal] & T_0 \arrow[r, equal] \arrow[d, equal] & \cdots \arrow[r, equal] & T_0 \arrow[d, equal] \\
B_1 \arrow[r, tail] & \cdots \arrow[r, tail] & B_i \arrow[r, tail] & T_0 \arrow[r, equal] & \cdots \arrow[r, equal] & T_0
\end{tikzcd}$$

$$* = \cdots = * = * = \cdots = * = S_0/S_0 \rightarrowtail S_1/S_0 \rightarrowtail \cdots \rightarrowtail S_n/S_0$$

$$B_1 \rightarrowtail \cdots \rightarrowtail B_i \rightarrowtail T_0 = \cdots = T_0 = T_0 \rightarrowtail T_1 \rightarrowtail \cdots \rightarrowtail T_n .$$

This shows that the hypothesis of the prop. on p. 245 is satisfied. The additivity theorem follows. //

The additivity theorem has several corollaries one of which is the theorem on p. 234. We refer to Waldhausen's paper in LNM 1126 for further reading.