2008.05.15

We have seen that the geometric realization functor has a right adjoint functor, and hence, preserves colimits. We will now show that the canonical map

$$|X[-]\times Y[-]| \longrightarrow |X[-]|\times|Y[-]|$$

is a continuous bijection. The inverse map need not be continuous, since the product topology has too few open sets. We will give the product a new topology such that also the inverse map is continuous. Next time, we will see that, from the point of view of homotopy theory, this change is immaterial. We first prove:

<u>Lemma</u> The canonical map

$$p: |\Delta[m][-]\times\Delta[n][-]| \to |\Delta[m][-]|\times|\Delta[n][-]|$$

is a homeomorphism.

<u>Proof</u> We construct the inverse map q. To do so, we need to

triangulate the product $\Delta[r] \times \Delta[s]$ of two topological standard simplices. More precisely, we claim that, given $(u,v) \in \Delta[r] \times \Delta[s]$, there exists unique morphisms $\theta_1 : [l] \to [r]$ and $\theta_2 : [l] \to [s]$ and a unique interior point $w \in \Delta[l]$ such that

$$(u,v) = (\theta_{1*}(w), \theta_{2*}(w))$$

Moreover, the map

$$\theta = (\theta_1, \theta_2) : [l] \to [r] \times [s]$$

is injective. To prove this, we let

$$\Delta'[k] = \{(x_1, \dots, x_k) \in \mathbb{R}^k \mid 0 \le x_1 \le \dots \le x_k \le 1\}$$

and note the homeomorphism

$$\varepsilon : \Delta[k] \to \Delta'[k]$$

defined by

$$x = \sum_{i \in [k]} a_i \cdot i \mapsto \varepsilon(x) = (x_1, \dots, x_k)$$

$$x_j = \sum_{1 \le i \le j} a_{i-1} \qquad (1 \le j \le k).$$

(The space $\Delta'[k]$ is called the configuration space of k ordered points in $[0,1]$.) We now write

$$\varepsilon(u) = (u_1, \dots, u_r)$$

$$\varepsilon(v) = (v_1, \dots, v_s)$$

and define $z \in \Delta[r+s]$ by

$$\varepsilon(z) = (z_1, z_2, \dots, z_{r+s}),$$

where each $z_i \in \{u_1, \dots, u_r, v_1, \dots, v_s\}$. So $z_1 \le \dots \le z_{r+s}$ is a reordering of the real numbers $u_1, \dots, u_r, v_1, \dots, v_s$ in increasing order. We define a map of partially ordered sets

$$\psi : [r+s] \longrightarrow [r] \times [s]$$

by $\psi(0) = (0,0)$ and, for $1 \le i \le r+s$,

$$\psi(i) = \begin{cases} \psi(i-1) + (1,0) & (z_i \in \{u_1, \dots, u_r\}) \\ \psi(i-1) + (0,1) & (z_i \in \{v_1, \dots, v_s\}) \end{cases}$$

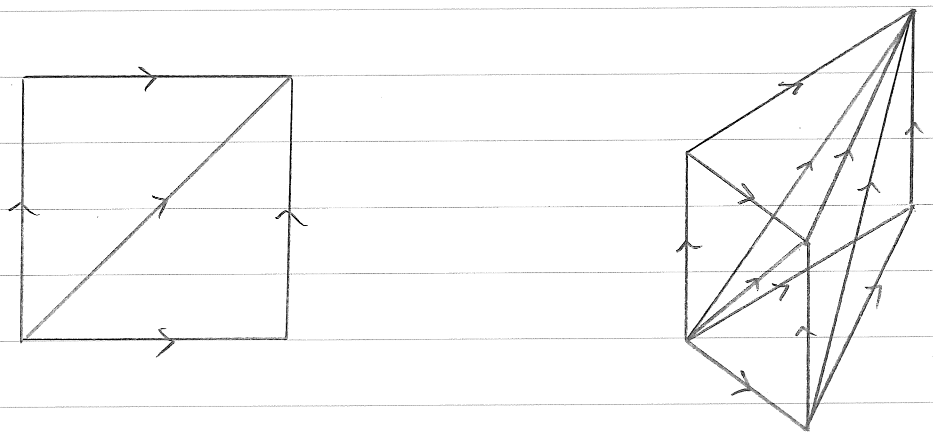
Note that $\psi(r+s) = (r,s)$. Then

$$\begin{array}{ccc} \Delta[r+s] & \xrightarrow{(\psi_{1*},\psi_{2*})} & \Delta[r]\times\Delta[s] \\ \cup & & \cup \\ z & \longmapsto & (u,v) \end{array}$$

We proved, in lecture 2, that there there is a unique order preserving map $\iota : [\ell] \to [r+s]$ and a unique interior point $w \in \Delta[\ell]$ such that $\iota_*(w) = z$. Now, the composite

$$\theta = \psi \circ \iota : [\ell] \to [r] \times [s].$$

is the desired map. This proves the claim. The triangulations of $\Delta[1]\times\Delta[1]$ and $\Delta[1]\times\Delta[2]$ look like this



We now define the map

$$q : |\Delta[m][-]| \times |\Delta[n][-]| \to |\Delta[m][-] \times \Delta[n][-]|$$

as follows: let

$$\xi = (\xi_1, \xi_2) \in |\Delta[m][-1]| \times |\Delta[n][-1]|,$$

and let (a, u), $a: [r] \hookrightarrow [m]$, $u \in \Delta[r]$ interior, and (b, v), $b: [s] \hookrightarrow [n]$, $v \in \Delta[s]$ interior, be the unique non-degenerate representatives of ξ_1 and ξ_2, respectively. The claim gives

$$\theta = (\theta_1, \theta_2): [\ell] \longrightarrow [r] \times [s]$$

and $w \in \Delta[\ell]$ interior such that $\theta_{1*}(w) = u$ and $\theta_{2*}(w) = v$. Then

$$q(\xi) := \text{class of } ((a, b) \circ \theta, w).$$

We show that q is inverse to p:

$$(pr_1 \circ q)(\xi) = \text{class of } pr_1((a, b) \circ \theta, w)$$

$$= \text{class of } (a \circ \theta_1, w)$$

$$= \text{class of } (a, \theta_{1*}(w))$$

$$= \text{class of } (a, u)$$

and, similarly,

$$(pr_2 \circ q)(\xi) = \text{class of } (b, v)$$

so $p \circ q = \mathrm{id}$. Next, let

$$\eta \in |\Delta[m][-] \times \Delta[n][-]|$$

have non-degenerate representative $((f,g), x)$, $(f,g) : [\ell] \hookrightarrow [m] \times [n]$, $x \in \Delta[\ell]$ interior, and factor

$$f : [\ell] \xrightarrow{\psi_1} [r] \xhookrightarrow{a} [m]$$

$$g : [\ell] \xrightarrow{\psi_2} [s] \xhookrightarrow{b} [n] .$$

Then

$$p(\eta) = (\text{class of } (f, x), \text{class of } (g, x))$$

$$= (\text{class of } (a, \psi_{1*}(x)), \text{class of } (b, \psi_{2*}(x))) .$$

But $x \in \Delta[\ell]$ is interior so

$$q(p(\eta)) := \text{class of } ((a,b) \circ \psi, x)$$

$$= \text{class of } ((f,g), x) .$$

Hence also $q \circ p = \mathrm{id}$, so p is a continuous bijection. Since the domain of p is Hausdorff and the target of p compact, it follows that p is a homeomorphism. //

If Y is a set, then the functor

$$- \times Y : \text{Sets} \longrightarrow \text{Sets}$$

has the right adjoint functor

$$\mathrm{Hom}_{\text{Sets}}(Y, -) : \text{Sets} \longrightarrow \text{Sets}.$$

The bijection

$$\mathrm{Hom}_{\text{Sets}}(X \times Y, Z) \xrightarrow{a} \mathrm{Hom}_{\text{Sets}}(X, \mathrm{Hom}_{\text{Sets}}(Y, Z))$$

is defined by

$$a(f)(x)(y) = f(x,y).$$

It follows that, for every diagram of sets, $X : I \longrightarrow \text{Sets}$, the canonical map

$$\operatorname*{colim}_I (X_i \times Y) \longrightarrow (\operatorname*{colim}_I X_i) \times Y$$

is a bijection. If Y is a topological space, the functor

$$- \times Y : \mathcal{T} \longrightarrow \mathcal{T}$$

in general does not have a right adjoint. We will instead consider a full subcategory $\mathcal{K} \subset \mathcal{T}$ where, for $Y \in \mathcal{K}$, the functor

$$- \times Y : \mathcal{K} \longrightarrow \mathcal{K}$$

does have a right adjoint. Let X be a topological space. We define the subset $U \subset X$ to be <u>compactly open</u> if, for every continuous map $f : K \to X$ with K compact and Hausdorff, $f^{-1}(U) \subset K$ is open. So open subsets $V \subset X$ are compactly open, but compactly open subsets $U \subset X$ are not necessarily open. We say that X is a <u>k-space</u> if every compactly open subset $U \subset X$ is an open subset and define $\mathcal{K}$ to be category whose objects are all k-spaces and whose morphisms are all continuous maps. The inclusion functor

$$i : \mathcal{K} \longrightarrow \mathcal{T}$$

has a right adjoint functor

$$k : \mathcal{T} \longrightarrow \mathcal{K}$$

that takes the topological space X to the k-space $k(X)$ defined to be the set X with the (new) topology where $U \subset k(X)$ is open if and only if $U \subset X$ is compactly open. It is clear that $k \circ i$ is the identity functor of $\mathcal{K}$. Since the functor $i : \mathcal{K} \to \mathcal{T}$ has a right adjoint, it preserves colimits. Hence, if $\mathcal{X} : I \to \mathcal{K}$ is a diagram of k-spaces, then the colimit

$$\operatorname*{colim}_I{}^{\mathcal{T}} (i \circ \mathcal{X})$$

is a k-space, and

$$i(\operatorname*{colim}_I{}^{\mathcal{K}} \mathcal{X}) = \operatorname*{colim}_I{}^{\mathcal{T}} (i \circ \mathcal{X}).$$

The functor $i : \mathcal{K} \to \mathcal{T}$, however, does not preserve limits. We define

$$\lim_{I}{}^{\mathcal{K}} \mathfrak{X} := k\left(\lim_{I}{}^{\mathcal{T}} (i \circ \mathfrak{X})\right);$$

This is the limit in $\mathcal{K}$ of the diagram $\mathfrak{X} : I \to \mathcal{K}$. In particular, the product in $\mathcal{K}$ of the two k-spaces X and Y is the k-space

$$X \times Y := k(i(X) \times i(Y)).$$

Now, if Y and Z are two k-spaces, we define the k-space

$$\underline{\mathrm{Hom}}_{\mathcal{K}}(Y, Z)$$

to be the set

$$\mathrm{Hom}_{\mathcal{K}}(Y, Z) := \mathrm{Hom}_{\mathcal{T}}(i(Y), i(Z))$$

with the k-space topology associated (by applying the functor k) to the topology with subbasis given by the set subsets

$$N(h, U) = \{ f : Y \to Z \mid f(h(K)) \subset U \},$$

where $h : K \to Y$ is continuous and K compact Hausdorff and $U \subset Z$ is open. The following holds:

Prop Let X, Y, and Z be k-spaces. Then the map

$$\underline{\mathrm{Hom}}_{\mathcal{K}}(X \times Y, Z) \xrightarrow{a} \underline{\mathrm{Hom}}_{\mathcal{K}}(X, \underline{\mathrm{Hom}}_{\mathcal{K}}(Y, Z))$$

defined by

$$a(f)(x)(y) = f(x, y)$$

is a homeomorphism

Proof This is proved in Gaunce Lewis' thesis. I have a copy for anyone interested. //

In particular, if Y is a k-space, then the functor

$$- \times Y : \mathcal{K} \longrightarrow \mathcal{K}$$

has the right adjoint functor

$$\underline{\mathrm{Hom}}_{\mathcal{K}}(Y, -) : \mathcal{K} \longrightarrow \mathcal{K}.$$

The geometric realization of the simplicial set $X[-]$ is a k-space. Indeed,

$$|X[-]| = \operatorname*{colim}_{\Delta / X[-]} \Delta[-]$$

and $\Delta[n]$ is compact, hence a k-space. From now on, we will always view geometric realization as a functor

$$|-| : \mathrm{Sets}^{\Delta^{op}} \longrightarrow \mathcal{K}.$$

The right-adjoint

$$\mathrm{Sin}(-)[-] : \mathcal{K} \longrightarrow \mathrm{Sets}^{\Delta^{op}}$$

is defined by

$$\mathrm{Sin}(Y)[-] = \mathrm{Hom}_{\mathcal{K}}(\Delta[-], Y).$$

<u>Thm</u> Let $X[-]$ and $Y[-]$ be simplicial sets. Then the canonical map

$$|X[-] \times Y[-]| \longrightarrow |X[-]| \times |Y[-]|$$

is an isomorphism in $\mathcal{K}$.

Proof We have the following series of canonical maps in $\mathcal{K}$:

$$|X[-] \times Y[-]|$$

$$\xrightarrow{①} \Big|\big(\operatorname*{colim}_{\Delta[m][-] \to X[-]} \Delta[m][-]\big) \times \big(\operatorname*{colim}_{\Delta[n][-] \to Y[-]} \Delta[n][-]\big)\Big|$$

$$\xleftarrow{②} \Big|\operatorname*{colim}_{\Delta[m][-] \to X[-]} \big(\operatorname*{colim}_{\Delta[n][-] \to Y[-]} (\Delta[m][-] \times \Delta[n][-])\big)\Big|$$

$$\xleftarrow{③} \Big|\operatorname*{colim}_{\substack{\Delta[m][-] \to X[-] \\ \Delta[n][-] \to Y[-]}} (\Delta[m][-] \times \Delta[n][-])\Big|$$

$$\xleftarrow{④} \operatorname*{colim}_{\substack{\Delta[m][-] \to X[-] \\ \Delta[n][-] \to Y[-]}} |\Delta[m][-] \times \Delta[n][-]|$$

$$\xrightarrow{⑤} \operatorname*{colim}_{\substack{\Delta[m][-] \to X[-] \\ \Delta[n][-] \to Y[-]}} (|\Delta[m][-]| \times |\Delta[n][-]|$$

$$\xrightarrow{⑥} \operatorname*{colim}_{\Delta[m][-] \to X[-]} \big(\operatorname*{colim}_{\Delta[n][-] \to Y[-]} (|\Delta[m][-]| \times |\Delta[n][-]|)\big)$$

$$\xrightarrow{⑦} \Big(\operatorname*{colim}_{\Delta[m][-]\to X[-]} |\Delta[m][-]|\Big) \times \Big(\operatorname*{colim}_{\Delta[n][-]\to Y[-]} |\Delta[n][-]|\Big)$$

$$\xrightarrow{⑧} |X[-]| \times |Y[-]| .$$

The maps ① and ⑧ are isomorphisms by a proposition proved in last lecture (p. 81). The maps ② and ⑦ are isomorphism because product with a simplicial set (resp. a k-space) commutes with colimits in $\mathrm{Sets}^{\Delta^{op}}$ (resp. in $\mathcal{K}$). The maps ③ and ⑥ are isomorphisms, since the colimit of a product diagram

$$\mathcal{X} \times \mathcal{Y} : I \times J \longrightarrow \mathcal{C}$$

is the product of the colimits of the two diagrams. (Exercise.) The map ④ is an isomorphism since geometric realization preserves colimits. Finally, the map ⑤ is an isomorphism by the lemma proved at the begining of the lecture. We leave as an exercise to show that the composition of these isomorphisms is the canonical map. //