A *pointed* k-space is a pair (X, x) of a k-space X and a point $x \in X$. The point x is called the base-point. A map $f : (X, x) \to (Y, y)$ is a continuous map $f : X \to Y$ such that $f(x) = y$. We write $\mathcal{K}_*$ for the category of pointed k-spaces. The forgetful functor $U : \mathcal{K}_* \to \mathcal{K}$ defined by $U(X, x) = X$ has a left-adjoint functor $(-)_+ : \mathcal{K} \to \mathcal{K}_*$ that to the k-space X associates the pointed k-space $(X_+, +)$, where $X_+ = X \sqcup \{+\}$ is the disjoint union of X and a base-point $+$. It follows that U preserves limits: The limit of the diagram $i \in I \mapsto (X_i, x_i) \in \mathcal{K}_*$ is

$$\lim_{i \in I} (X_i, x_i) := \Big(\lim_{i \in I} X_i, (x_i) \Big).$$

The functor $(-)_+$ preserves colimits, but the functor U does not. The colimit of $i \in I \mapsto (X_i, x_i)$ is the quotient space

$$\operatorname*{colim}_{i \in I} (X_i, x_i) := \Big(\big(\operatorname*{colim}_{i \in I} X_i\big) / B, \bar{B} \Big)$$

obtained from the colimit of the

diagram $i \in I \mapsto X_i \in \mathcal{K}$ by collapsing the subspace

$$B = \{ in_i(x_i) \mid i \in I \}$$

to a single point $\overline{B}$. The point $\overline{B}$ is the base-point of the colimit. For example, the coproduct of (X,x) and (Y,y) is the quotient space

$$(X,x) \vee (Y,y) := \left((X \amalg Y)/\{x,y\}, \overline{\{x,y\}} \right)$$

obtained from $X \amalg Y$ by identifying $x \in X$ and $y \in Y$ to a single point $\overline{\{x,y\}}$. The coproduct is called the <u>wedge sum</u> of (X,x) and (Y,y). There is a canonical map

$$(X,x) \vee (Y,y) \xrightarrow{h} (X,x) \times (Y,y)$$

defined by the diagram

$$\begin{tikzcd}
 & (X,x) \vee (Y,y) \arrow[dl, "f"'] \arrow[d, "h"] \arrow[dr, "g"] & \\
(X,x) & (X,x) \times (Y,y) \arrow[l, "pr_1"'] \arrow[r, "pr_2"] & (Y,y)
\end{tikzcd}$$

where the map f collapses Y to the base-point and the map g collapses X to the base-point. We define the smash product of (X,x) and (Y,y) to be the pointed k-space

$$(X,x) \wedge (Y,y)$$

$$= \left(X \times Y \Big/ h(X \vee Y),\ \overline{h(X \vee Y)} \right)$$

$$= \left(\frac{X \times Y}{\{x\} \times Y \cup X \times \{y\}},\ \overline{\{x\} \times Y \cup X \times \{y\}} \right)$$

obtained from $X \times Y$ by collapsing the subspace $h(X \vee Y)$ to a point. We also define the mapping space

$$\underline{\mathrm{Hom}}_{\mathcal{K}_*}((X,x),(Y,y)) \subset \underline{\mathrm{Hom}}_{\mathcal{K}}(X,Y)$$

to be the subspace of all maps $f : X \to Y$ such that $f(x) = y$ with the constant map $\bar{y} : (X,x) \to (Y,y)$, $\bar{y}(x') = y$, as the base-point. The canonical homeomorphism

$$\underline{\mathrm{Hom}}_{\mathcal{K}}(X \times Y, Z) \xrightarrow{a} \underline{\mathrm{Hom}}_{\mathcal{K}}(X, \underline{\mathrm{Hom}}_{\mathcal{K}}(Y,Z))$$

induces the homeomorphism

$$\underline{\mathrm{Hom}}_{\mathcal{T}_*}((X,x)\wedge(Y,y),(Z,z))$$

$$\xrightarrow{a} \underline{\mathrm{Hom}}_{\mathcal{T}_*}((X,x),\underline{\mathrm{Hom}}_{\mathcal{T}_*}((Y,y),(Z,z))).$$

defined by

$$a(f)(x')(y') = f(\text{class of } (x',y')).$$

In particular, the functor $-\wedge(Y,y)$ is left adjoint to $\underline{\mathrm{Hom}}_{\mathcal{T}_*}((Y,y),-)$.

We define the n-sp<u>here</u> to be the pointed k-space

$$(S^n,\infty) = ((\mathbb{R}^n)^+,\infty)$$

given by the one-point compactification of $\mathbb{R}^n$ with the point ∞ as the base-point. So S^n is the set $\mathbb{R}^n \amalg \{\infty\}$ and the subset $U \subset S^n$ is open iff either $U \subset \mathbb{R}^n$ is open or $\infty \in U$ and $S^n \setminus U = K$ with $K \subset \mathbb{R}^n$ compact. There is a canonical homeomorphism

$$(S^{m+n}, \infty) \xrightarrow{\sim} (S^m, \infty) \wedge (S^n, \infty)$$

that takes $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$ to the class of (x, y) and the base-point ∞ to the base-point $\infty \times S^n \cup S^m \times \infty$.

Let $(X, *)$ be a pointed k-space. We define the suspension and loop space of $(X, *)$ by

$$\Sigma(X, *) = (X, *) \wedge (S^1, \infty)$$

$$\Omega(X, *) = \underline{\mathrm{Hom}}_{\mathcal{T}_*}((S^1, \infty), (X, *)).$$

Then Σ is left adjoint to Ω and we have canonical homeomorphisms

$$(S^{n+1}, \infty) \xrightarrow{\sim} \Sigma(S^n, \infty).$$

A homotopy from $f: (X, *) \to (Y, y)$ to $g: (X, *) \to (Y, y)$ is a commutative diagram

$$\begin{tikzcd}
(X,x) \arrow[d, "\iota_0"] \arrow[drr, "f"] & & \\
(X,x) \wedge ([0,1]_+, +) \arrow[rr, "h"] & & (Y,y) \\
(X,x) \arrow[u, "\iota_1"'] \arrow[urr, "g"'] & &
\end{tikzcd}$$

where $\iota_\varepsilon(x') =$ class of (x', ε). If a homotopy from f to g exists, we say that f and g are <u>homotopic</u> and write $f \sim g$. This defines an equivalence relation on the set of maps from (X,x) to (Y,y), and we define

$$[(X,x),(Y,y)] = \mathrm{Hom}_{\mathcal{T}_*}((X,x),(Y,y))/\sim$$

to be the pointed set of equivalence classes. The base-point of this set is the class of the constant map $\bar{y} : (X,x) \to (Y,y)$. If $f : (X,x) \to (Y,y)$ is homotopic to $\bar{y} : (X,x) \to (Y,y)$, we say that f is <u>null</u> - <u>homotopic</u>.

We consider the maps

$$\psi : (S^1, \infty) \longrightarrow (S^1, \infty) \vee (S^1, \infty)$$

$$\iota : (S^1, \infty) \longrightarrow (S^1, \infty)$$

defined by

$$\psi(t) = \begin{cases} in_1(\log t) & (t \in (0, \infty)) \\ in_2(-\log(-t)) & (t \in (-\infty, 0)) \\ \infty & (t \in \{0, \infty\}) \end{cases}$$

$$\iota(t) = \begin{cases} -t & (t \in \mathbb{R}) \\ \infty & (t = \infty) \end{cases}.$$

The following diagrams in $\mathcal{K}_*$ are homotopy commutative

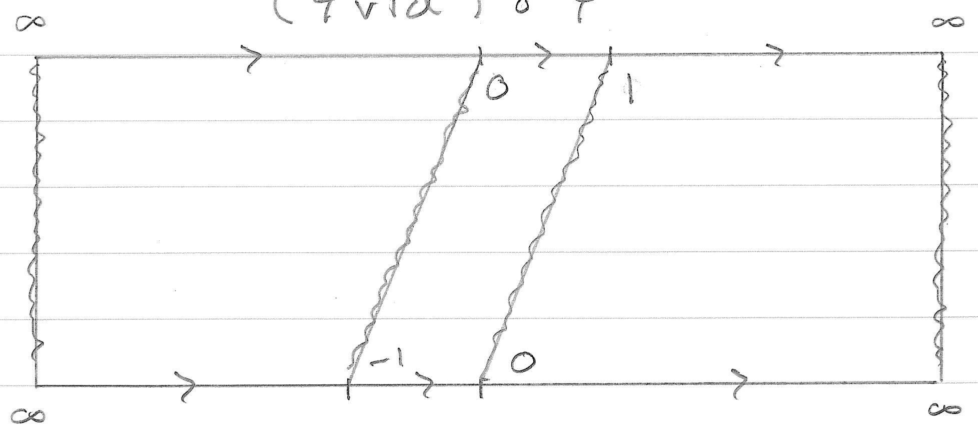
$$\begin{tikzcd} (S^1, \infty) \arrow[r, "\psi"] \arrow[d, "\psi"] & (S^1, \infty) \vee (S^1, \infty) \arrow[d, "\psi \vee id"] \\ (S^1, \infty) \vee (S^1, \infty) \arrow[r, "id \vee \psi"] & (S^1, \infty) \vee (S^1, \infty) \vee (S^1, \infty) \end{tikzcd}$$

$$\begin{tikzcd}
 & (S^1,\infty) \arrow[dl, "\mathrm{id}"'] \arrow[d, "\psi"] \arrow[dr, "\mathrm{id}"] & \\
(S^1,\infty) & (S^1,\infty)\vee(S^1,\infty) \arrow[l, "\overline{\infty}+\mathrm{id}"'] \arrow[r, "\mathrm{id}+\overline{\infty}"] & (S^1,\infty)
\end{tikzcd}$$

$$\begin{tikzcd}
 & (S^1,\infty)\vee(S^1,\infty) \arrow[r, "\iota\vee\mathrm{id}"] & (S^1,\infty)\vee(S^1,\infty) \arrow[dr, "\nabla"] & \\
(S^1,\infty) \arrow[ur, "\psi"] \arrow[dr, "\psi"'] \arrow[rrr, "\overline{\infty}"] & & & (S^1,\infty) \\
 & (S^1,\infty)\vee(S^1,\infty) \arrow[r, "\mathrm{id}\vee\iota"] & (S^1,\infty)\vee(S^1,\infty) \arrow[ur, "\nabla"'] &
\end{tikzcd}$$

The following picture explains the homotopy between the two compositions in the first diagram:

$(\psi\vee\mathrm{id})\circ\psi$



$(\mathrm{id}\vee\psi)\circ\psi$

We define the <u>fundamental</u> <u>group</u> of the pointed k-space (X,x) to be the set

$$\pi_1(X,x) = [(S^1,\infty),(X,x)]$$

with the following group structure: The product of the classes of the maps $f:(S^1,\infty)\to(X,x)$ and $g:(S^1,\infty)\to(X,x)$ is the class of the map

$$f*g:(S^1,\infty)\longrightarrow(X,x)$$

defined to be the composition

$$(S^1,\infty)\xrightarrow{\nu}(S^1,\infty)\vee(S^1,\infty)\xrightarrow{f+g}(X,x);$$

the inverse of the class of f is the class of the composite map

$$(S^1,\infty)\xrightarrow{c}(S^1,\infty)\xrightarrow{f}(X,x);$$

and the unit element $e\in\pi_1(X,x)$ is the class of the constant map $\bar{x}$.

Similarly, we can define two multiplications $*$ and $*'$ on the pointed set

$$\pi_2(X,x) = [(S^2,\infty),(X,x)]$$
$$\xleftarrow{\sim} [(S^1,\infty)\wedge(S^1,\infty),(X,x)]$$

To define $*$, we first recall that the functor $-\wedge(Z,z)$ has a right adjoint, and hence, preserves colimits. In particular, the canonical map

$$(X,x)\wedge(Z,z) \vee (Y,y)\wedge(Z,z)$$
$$\longrightarrow ((X,x)\vee(Y,y))\wedge(Z,z)$$

is a homeomorphism. We write

$$\rho\colon ((X,x)\vee(Y,y))\wedge(Z,z)$$
$$\longrightarrow (X,x)\wedge(Z,z) \vee (Y,y)\wedge(Z,z)$$

for the inverse homeomorphism and call it the right distributivity homeomorphism. Now, let

$$f,g\colon (S^1,\infty)\wedge(S^1,\infty) \longrightarrow (X,x)$$

be two maps. Then we define

$$f * g : (S^1,\infty)\wedge(S^1,\infty) \longrightarrow (X,*)$$

to be the composite map

$$(S^1,\infty)\wedge(S^1,\infty)$$

$$\xrightarrow{\nu\wedge \mathrm{id}} ((S^1,\infty)\vee(S^1,\infty))\wedge(S^1,\infty)$$

$$\xrightarrow{\rho} (S^1,\infty)\wedge(S^1,\infty)\vee(S^1,\infty)\wedge(S^1,\infty)$$

$$\xrightarrow{f+g} (X,*).$$

The product $*$ is the induced map of homotopy classes of maps.

To define $*'$, we first define

$$\gamma : (X,*)\wedge(Y,y) \longrightarrow (Y,y)\wedge(X,*)$$

to be the homeomorphism given by

$$\gamma(\text{class of } (x',y')) = \text{class of } (y',x').$$

We then define the left distributivity homeomorphism

$$\lambda : (X,x) \wedge ((Y,y) \vee (Z,z))$$
$$\longrightarrow (X,x)\wedge(Z,z) \vee (X,x)\wedge(Y,y)$$

to be the composition

$$(X,x) \wedge ((Y,y) \vee (Z,z))$$
$$\xrightarrow{\gamma} ((Y,y)\vee(Z,z))\wedge(X,x)$$
$$\xrightarrow{\rho} (Y,y)\wedge(X,x) \vee (Z,z)\wedge(X,x)$$
$$\xrightarrow{\gamma\vee\gamma} (X,x)\wedge(Y,y) \vee (X,x)\wedge(Z,z).$$

Then with f and g as before, we define the map

$$f *' g : (S^1,\infty)\wedge(S^1,\infty) \longrightarrow (X,x)$$

to be the composition

$$(S^1,\infty)\wedge(S^1,\infty)$$
$$\xrightarrow{\mathrm{id}\wedge\psi} (S^1,\infty)\wedge((S^1,\infty)\vee(S^1,\infty))$$
$$\xrightarrow{\lambda} (S^1,\infty)\wedge(S^1,\infty)\vee(S^1,\infty)\wedge(S^1,\infty)$$
$$\xrightarrow{f+g} (X,x).$$

The commutativity of the diagram on the next page shows that $*$ and $*'$ satisfy the identity

$$(f *' g) * (f' *' g') = (f * f') *' (g * g').$$

Indeed, the left-hand side is equal to the composition of the maps on the left-hand side of the diagram and the right-hand side is equal to the composition of the maps on the right-hand side of the diagram. We have omited to write base-points in the diagram. The sub-diagrams ① and ② commute by naturality of the maps ρ and λ, respectively. The homeomorphism τ is defined by

$$\begin{tikzcd}
(X,x) \arrow[r, "in_1"] \arrow[dr, "in_2"'] & (X,x) \vee (Y,y) \arrow[d, "\tau"] & (Y,y) \arrow[l, "in_2"'] \arrow[dl, "in_1"] \\
& (Y,y) \vee (X,x) &
\end{tikzcd}$$

One checks directly that ③ commutes. It is clear that ④ commutes. The under-bar has no mathematical meaning.

$$
\begin{tikzcd}
(S^1 \vee S^1) \wedge \underline{S}^1 \arrow[d, "\rho"] & & S^1 \wedge \underline{S}^1 \arrow[ll, "\psi \wedge id"'] \arrow[rr, "id \wedge \psi"] \arrow[d, "\psi \wedge \psi"] & & S^1 \wedge (\underline{S}^1 \vee \underline{S}^1) \arrow[d, "\lambda"] \\
S^1 \wedge \underline{S}^1 \vee S^1 \wedge \underline{S}^1 \arrow[dr, "id \wedge \psi \vee id \wedge \psi"] \arrow[rr, phantom, "1"] & & (S^1 \vee S^1) \wedge (\underline{S}^1 \vee \underline{S}^1) \arrow[dl, "\rho", "\sim"'] \arrow[dr, "\lambda", "\sim"'] \arrow[rr, phantom, "2"] & & S^1 \wedge \underline{S}^1 \vee S^1 \wedge \underline{S}^1 \arrow[dl, "\psi \wedge id \vee \psi \wedge id"'] \\
& S^1 \wedge (\underline{S}^1 \vee \underline{S}^1) \vee S^1 \wedge (\underline{S}^1 \vee \underline{S}^1) \arrow[d, "\lambda \vee \lambda", "\sim"'] \arrow[rr, phantom, "3"'] & & (S^1 \vee S^1) \wedge \underline{S}^1 \vee (S^1 \vee S^1) \wedge \underline{S}^1 \arrow[d, "\rho \vee \rho", "\sim"'] & \\
& S^1 \wedge \underline{S}^1 \vee S^1 \wedge \underline{S}^1 \vee S^1 \wedge \underline{S}^1 \vee S^1 \wedge \underline{S}^1 \arrow[rr, "id \vee \tau \vee id"] \arrow[rr, phantom, "4"', shift right=4ex] \arrow[dr, "f+g+f'+g'"'] & & S^1 \wedge \underline{S}^1 \vee S^1 \wedge \underline{S}^1 \vee S^1 \wedge \underline{S}^1 \vee S^1 \wedge \underline{S}^1 \arrow[dl, "f+f'+g+g'"] & \\
& & X & &
\end{tikzcd}
$$

Let us write $[f]$ for the homotopy class of the map f, and let $e = \bar{x}$ be the constant map. The formula on page 113 shows that

$$[f*g] = [(f*'e)*(e*'g)]$$

$$= [(f*e)*'(e*g)] = [f*'g].$$

and

$$[g*f] = [(e*'g)*(f*'e)]$$

$$= [(e*f)*'(g*e)] = [f*'g].$$

Hence, $*$ and $*'$ define the same group structure on $\pi_2(X,x)$ and this group structure is abelian. For all $n \geq 0$, we define

$$\pi_n(X,x) = [(S^n,\infty),(X,x)].$$

It is a pointed set, if $n=0$, a group, if $n=1$, and an abelian group, if $n \geq 2$. We define

$$\pi_n((\Omega(X,x),\bar{x}) \xrightarrow{v} \pi_{n+1}(X,x)$$

to be the isomorphism given by the composition

$$[(S^n,\infty), \Omega(X,x)]$$
$$\xleftarrow{\sim} [\Sigma(S^n,\infty), (X,x)]$$
$$\xrightarrow{\sim} [(S^{n+1},\infty), (X,x)]$$

of the inverse of the adjunction isomorphism and the canonical isomorphism.

Let $f : (X,x) \to (Y,y)$ be a map. Then we have the induced map

$$f_* : \pi_n(X,x) \longrightarrow \pi_n(Y,y)$$

defined by

$$f_*([\omega]) = [f \circ \omega].$$

We will show that f_* fits in a long-exact sequence of homotopy groups. We first define the <u>path space</u> of (Y,y) to be the pointed mapping space

$$(P(Y,y),\bar{y}) = (\underline{\mathrm{Hom}}_{\mathcal{T}_*}(([0,1],0),\bar{y})).$$

It is a contractible space. Indeed, a homotopy from the constant map $\bar{y}$ to the identity map is given by

$$h : (P(Y,y),\bar{y}) \wedge ([0,1],0) \to (P(Y,y),\bar{y})$$

$$h(\sigma,s)(t) = \sigma(s\cdot t).$$

We now define the <u>mapping fiber</u> of f over y to be the pull-back

$$\begin{array}{ccc} (F(f,y),(x,\bar{y})) & \xrightarrow{i} & (X,x) \\ \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle f} \\ (P(Y,y),\bar{y}) & \xrightarrow{ev_1} & (Y,y) \end{array}$$

where $ev_1(\sigma) = \sigma(1)$.

<u>Lemma</u> The sequence

$$\pi_n(F(f,y),(x,\bar{y})) \xrightarrow{i_*} \pi_n(X,x) \xrightarrow{f_*} \pi_n(Y,y)$$

is exact.

Proof The composition $f_* \circ i_*$ is equal to the composition

$$\pi_n(F(f,y),(x,\bar{y})) \xrightarrow{i'_*} \pi_n(P(Y,y),\bar{y}) \xrightarrow{w_{1*}} \pi_n(Y,y)$$

and $\pi_n(P(Y,y),\bar{y}) = \{[\bar{y}]\}$. Hence, $f_* \circ i_*$ is the constant map $[\bar{y}]$.

Let $[\omega] \in \pi_n(X,x)$ and suppose that $f_*([\omega]) = [\bar{y}]$. Then there exists a homotopy

$$h : (S^n,\infty) \wedge ([0,1],0) \longrightarrow (Y,y)$$

from the constant map $\bar{y}$ to the map $f \circ \omega$. Hence, we have a commutative diagram

$$\begin{array}{ccc} (S^n,\infty) & \xrightarrow{\omega} & (X,x) \\ \downarrow i_1 & & \downarrow f \\ (S^n,\infty) \wedge ([0,1],0) & \xrightarrow{h} & (Y,y) \end{array}$$

We adjoin h to get the map

$$a(h) : (S^n,\infty) \longrightarrow (P(Y,y),\bar{y})$$

Then the diagram

$$\begin{array}{ccc} (S^n,\infty) & \xrightarrow{\omega} & (X,x) \\ \Big\downarrow a(h) & & \Big\downarrow f \\ (P(Y,y),\bar{y}) & \xrightarrow{ev_1} & (Y,y) \end{array}$$

commutes. The induced map

$$(\omega, a(h)) : (S^n,\infty) \longrightarrow (F(f,y),(x,\bar{y}))$$

satisfies $i\circ(\omega,a(h)) = \omega$. Hence, $i_*([(\omega,a(h))]) = [\omega]$, so the kernel of f_* is equal to the image of i_*. //

We may apply the lemma to the map

$$i : (F(f,y),(x,\bar{y})) \to (X,x).$$

It turns out that the mapping fiber $(F(i,x),((x,\bar{y}),\bar{x}))$ is homotopy equivalent to the loop space $(\Omega(Y,y),\bar{y})$. To see this, we first let

$$\tau : ([-1,1]/\{-1,1\}, \overline{\{-1,1\}}) \to (S^1,\infty)$$

be the homeomorphism defined by

$$\tau(t) = \tan\left(\frac{\pi}{2} \cdot t\right).$$

It induces the homeomorphism

$$(\Omega(Y, y), \bar{y}) = \left(\underline{\mathrm{Hom}}_{\mathcal{K}_*}((S^1, \infty), (Y, y)), \bar{y}\right)$$

$$\xrightarrow[\sim]{\tau^*} \left(\underline{\mathrm{Hom}}_{\mathcal{K}_*}(([-1,1]/\{-1,1\}, \overline{\{-1,1\}}), (Y, y)), \bar{y}\right).$$

Now, we have the pull-back diagram

$$\begin{tikzcd}
(\Omega(Y, y), \bar{y}) \arrow[r, "\rho_-"] \arrow[d, "\rho_+"] & (P(Y, y), \bar{y}) \arrow[d, "ev_1"] \\
(P(Y, y), \bar{y}) \arrow[r, "ev_1"] & (Y, y)
\end{tikzcd}$$

where

$$\rho_-(\omega)(t) = \tau^*(\omega)(t-1), \quad t \in [0,1],$$

$$\rho_+(\omega)(t) = \tau^*(\omega)(1-t), \quad t \in [0,1].$$

On the other hand, the mapping fiber of i over $(x, \bar{y})$ is defined to be the pull-back

$$\begin{array}{ccc}
(F(i,x),((x,\bar{y}),\bar{x})) & \xrightarrow{j} & (F(f,y),(x,\bar{y})) \\
\downarrow j' & & \downarrow i \\
(P(X,x),\bar{x}) & \xrightarrow{ev_1} & (X,x)
\end{array}$$

and the mapping fiber of f over y is defined to be the pull-back

$$\begin{array}{ccc}
(F(f,y),(x,\bar{y})) & \xrightarrow{i'} & (P(Y,y),\bar{y}) \\
\downarrow i & & \downarrow ev_1 \\
(X,x) & \xrightarrow{f} & (Y,y)
\end{array}$$

It follows that we have a pull-back

$$\begin{array}{ccc}
(F(i,x),((x,\bar{y}),\bar{x})) & \xrightarrow{i'\circ j} & (P(Y,y),\bar{y}) \\
\downarrow j' & & \downarrow ev_1 \\
(P(X,x),\bar{x}) & \xrightarrow{f\circ ev_1} & (Y,y)
\end{array}$$

We now define

$$\alpha : (F(i,x),((x,\bar{y}),\bar{x})) \longrightarrow (\Omega(Y,y),\bar{y})$$

to be the unique map such that

$$\rho_- \circ \alpha = i' \circ j$$

$$\rho_+ \circ \alpha = f_* \circ j'$$

where $f_* : (P(X,x), \bar{x}) \to (P(Y,y), \bar{y})$ is the map induced by f. We define

$$\beta : (\Omega(Y,y), \bar{y}) \to (F(i,x), ((x,\bar{y}), \bar{x}))$$

to be the unique map such that

$$(i' \circ j \circ \beta)(\omega)(t) = \tau^*(\omega)(2t-1), \quad t \in [0,1]$$

$$(j' \circ \beta)(\omega) = \bar{x}.$$

There are homotopies

$$\alpha \circ \beta \simeq \mathrm{id}$$

$$\beta \circ \alpha \simeq \mathrm{id}.$$

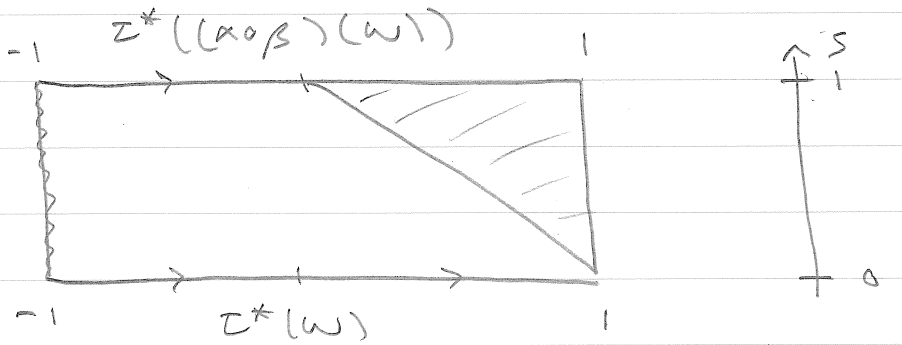
We give the first homotopy and leave it as an exercise to write down the second homotopy. We have

$$\tau^*((\alpha\circ\beta)(\omega))(t) = \begin{cases} \tau^*(\omega)(2t+1) & (-1 \le t \le 0) \\ \eta & (0 \le t \le 1) \end{cases}$$

so a homotopy h from $\tau^*(\omega)$ to $\tau^*((\alpha\circ\beta)(\omega))$ is given by

$$h(\tau^*(\omega), s)(t) = \begin{cases} \tau^*(\omega)(st+s+t) & \left(-1 \le t \le \frac{1-s}{1+s}\right) \\ \eta & \left(\frac{1-s}{1+s} \le t \le 1\right) \end{cases}$$

The following picture illustrates h:



We define the boundary map

$$\partial : (\Omega(Y,\eta), \bar{\eta}) \longrightarrow (F(f,\eta), (x, \bar{\eta}))$$

to be the composition $j \circ \beta$. Then the lemma shows that the maps

$$(\Omega(Y,\eta),\bar{\eta}) \xrightarrow{\partial} (F(f,\eta),(x,\bar{\eta})) \xrightarrow{i} (X,x) \xrightarrow{f} (Y,\eta)$$

induce an exact sequence

$$\pi_n(\Omega(Y,y),\bar{y}) \xrightarrow{\partial_*} \pi_n(F(f,y),(x,\bar{y})) \xrightarrow{i_*} \pi_n(X,x) \xrightarrow{f_*} \pi_n(Y,y).$$

of homotopy groups. Hence, we have the exact sequence

$$\pi_{n+1}(Y,y) \xrightarrow{\delta} \pi_n(F(f,y),(x,\bar{y})) \xrightarrow{i_*} \pi_n(X,x) \xrightarrow{f_*} \pi_n(Y,y)$$

where δ is the composite

$$\pi_{n+1}(Y,y) \underset{\approx}{\xrightarrow{\tilde{\sigma}^{-1}}} \pi_n(\Omega(Y,y),\bar{y}) \xrightarrow{\partial_*} \pi_n(F(f,y),(x,\bar{y})).$$

We leave it as an exercise to show that the composition

$$(\Omega(X,x),\bar{x}) \xrightarrow{\partial'} (F(i,x),((x,\bar{y}),\bar{x})) \underset{\sim}{\xrightarrow{\alpha}} (\Omega(Y,y),\bar{y})$$

of the boundary map ∂' corresponding to the map i and the homotopy equivalence α is homotopic to the composition

$$(\Omega(X,x),\bar{x}) \xrightarrow{\iota^*} (\Omega(X,x),\bar{x}) \xrightarrow{f_*} (\Omega(Y,y),\bar{y})$$

of the map induced by the inversion ι of the circle and the map induced by f. It follows that we have a long-exact sequence of homotopy groups

$$
\begin{tikzcd}
\vdots \arrow[d] \\
\pi_{n+2}(Y, y) \arrow[d, "-\delta"] \\
\pi_{n+1}(F(f,y), (x, \bar{y})) \arrow[d, "-i_*"] \\
\pi_{n+1}(X, x) \arrow[d, "-f_*"] \\
\pi_{n+1}(Y, y) \arrow[d, "\delta"] \\
\pi_n(F(f,y), (x, \bar{y})) \arrow[d, "i_*"] \\
\pi_n(X, x) \arrow[d, "f_*"] \\
\pi_n(Y, y) \arrow[d, "-\delta"] \\
\vdots
\end{tikzcd}
$$