Let G be a group. We will also write G for the category with a single object $*$ and with

$$\mathrm{Hom}_G(*,*) = G.$$

The composition of two morphisms $g, g' : * \to *$ is defined by

$$g \circ g' := gg'.$$

We will calculate the homotopy groups classifying space

$$BG = |NG[-]|.$$

We define a second category $T(G)$ associated with the group G:

$$\mathrm{ob}\, T(G) = G,$$

and for every $g, g' \in G$, there is a unique morphism from g to g'. Hence, every object of $T(G)$ is both an initial and a terminal object. It follows that the space

$$EG := |NT(G)[-1]|$$

is contractible. There is a functor

$$p : T(G) \longrightarrow G$$

that takes $g \in \mathrm{ob}\, T(G)$ to $* \in \mathrm{ob}\, G$ and takes the unique morphism from g_0 to g_1 in $T(G)$ to the morphism $g_1 g_0^{-1}$ from $*$ to $*$ in G. The functor p induces a map

$$p : EG \longrightarrow BG,$$

and hence, a long-exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_{n+1}(EG, e) \xrightarrow{p_*} \pi_{n+1}(BG, *) \xrightarrow{\delta}$$

$$\pi_n(F(p,*), (e, \bar{*})) \longrightarrow \pi_n(EG, e) \xrightarrow{p_*} \cdots$$

Since EG is contractible, the group $\pi_n(EG, e)$ is zero, for all $n \geq 0$. (For $n = 0$, this is just a pointed set with a single element.) follows that the map

$$\delta : \pi_{n+1}(BG, *) \longrightarrow \pi_n(F(p,*), (e, \bar{*}))$$

is an isomorphism, for all $n \geq 0$. (For $n = 0$, the statement is that the group $\pi_1(BG, *)$ acts freely and transitively on $\pi_0(F(p,*), (e,\bar{*}))$.) We must understand the mapping fiber $F(p,*)$.

We consider the map

$$NT(G)[n] \times G \xrightarrow{\mu} NT(G)[n]$$

that takes functor $f : [n] \to T(G)$ to the functor $f \cdot g : [n] \to T(G)$ defined by

$$(f \cdot g)(i) = f(i)g$$

for all $i \in ob[n]$. Here $f(i) \in ob\,T(G) = G$ and $f(i)g$ is the product in the group G. The map μ defines a right action by G on $NT(G)[n]$. It is clearly a free action, and the map p induces a bijection

$$\bar{p} : NT(G)[n]/G \xrightarrow{\sim} NG[n]$$

from the set of orbit for the action by G on $NT(G)[n]$ onto $NG[n]$. For every morphism $\theta : [m] \to [n]$, the diagram

$$\begin{tikzcd} NT(G)[n] \times G \arrow[r, "\mu"] \arrow[d, "\theta^* \times id"] & NT(G)[n] \arrow[d, "\theta^*"] \\ NT(G)[m] \times G \arrow[r, "\mu"] & NT(G)[m] \end{tikzcd}$$

commutes. So G acts on the simplicial set $NT(G)[-]$, and

$$\bar{p} : NT(G)[-]/G \longrightarrow NG[-]$$

is an isomorphism of simplicial sets. To take orbits for a group action is a colimit and geometric realization preserves colimits. Hence, G acts on EG and the map

$$p : EG \longrightarrow BG$$

indices a homeomorphism

$$\bar{p} : (EG)/G \xrightarrow{\sim} BG.$$

<u>Prop</u> The map $p : EG \to BG$ is a covering space.

<u>Proof</u> Let $Y[-] \subset NG[-]$ be a sub-simplicial set, and let $X[-] \subset NT(G)[-]$ be the pull-back

$$\begin{array}{ccc} X[-] & \hookrightarrow & NT(G)[-] \\ \downarrow p & & \downarrow p \\ Y[-] & \hookrightarrow & NG[-] \end{array}.$$

Then G acts freely on $X[-]$ and p induces an isomorphism of $X[-]/G$ onto $Y[-]$. We first show that

$$p : |X[-]| \to |Y[-]|$$

is a covering space assuming that $Y[-]$ has only finitely many non-degenerate simplices. The proof is by induction on the number N of non-degenerate simplices in $Y[-]$. If $N = 0$, then $Y[-]$ and $X[-]$ are both the empty simplicial set, so the statement is trivially true. To prove the induction step, let $y \in Y[n]$ be non-degenerate with n maximal, and let $\sigma_y \colon \Delta[n][-] \to Y[-]$ be the corresponding map from the standard n-simplex. Let also $\partial\Delta[n][-] \subset \Delta[n][-]$ be the boundary simplicial set where $\partial\Delta[n][m]$ is the set of maps $\theta \colon [m] \to [n]$ that are <u>not</u> surjective. Finally, let $Z[-] \subset Y[-]$ be the sub-simplicial set generated by the remaining non-degenerate simplices of $Y[-]$. Every element of $Z[m] \subset Y[m]$ can be written as $z = \theta^* y'$ where $y' \in Y[k]$ is non-degenerate and different from y, and $\theta \colon [m] \to [k]$ is a map in Δ. We claim that $\sigma_y \colon \Delta[n][-] \to Y[-]$ maps $\partial\Delta[n][-]$ to $Z[-]$. Indeed, the non-degenerate simplices of $\partial\Delta[n][-]$ all belong to $\partial\Delta[n][k]$ with $k < n$. Hence, every element of $x \in \partial\Delta[n][m]$ can be written as $x = \theta^* x'$, for some $x' \in \partial\Delta[n][k]$ with $k < n$. It follows that

$$\sigma_y(x) = \sigma_y(\theta^* x') = \theta^*(\sigma_y(x')).$$

The element $\sigma_y(x') \in Y[k]$ may or may not be non-degenerate. In either case, we can write $\sigma_y(x')$ uniquely as $\sigma_y(x') = \psi^* y'$ with $y' \in Y[\ell]$ non-degenerate and $\psi : [k] \to [\ell]$. Then $\ell \leq k < n$, which shows that

$$\sigma_y(x) = \theta^*(\sigma_y(x')) = (\psi \circ \theta)^*(y')$$

is in $Z[m]$ as claimed. It follows that we have a push-out diagram of simplicial sets

$$\begin{array}{ccc} \partial\Delta[n][-] & \xrightarrow{\sigma_y} & Z[-] \\ \big\downarrow & & \big\downarrow \\ \Delta[n][-] & \xrightarrow{\sigma_y} & Y[-] \end{array},$$

The geometric realization of this diagram is a push-out diagram of k-spaces. One can show that the canonical homeomorphism

$$\Delta[n] \xrightarrow{\sim} |\Delta[n][-]|$$

restricts to a homeomorphism

$$\partial\Delta[n] \xrightarrow{\sim} |\partial\Delta[n][-]|$$

where $\partial\Delta[n] \subset \Delta[n]$ is the boundary which consists of the points

$$a = \sum_{i \in [n]} a_i \cdot i$$

with $a_i = 0$ for some $i \in [n]$. So we get a push-out square of k-spaces

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "\sigma_y"] \arrow[d, hook] & {|Z[-1]|} \arrow[d, hook] \\ \Delta[n] \arrow[r, "\sigma_y"] & {|Y[-1]|} \end{tikzcd}.$$

Now, for every $v \in |Y[-1]|$, we must find $v \in V \subset |Y[-1]|$ open and a commutative diagram

$$\begin{tikzcd} V \times G \arrow[r, "\varphi"] \arrow[d, "pr_1"] & p^{-1}(V) \arrow[d, "p"] \\ V \arrow[r, equal] & V \end{tikzcd}$$

with φ a G-equivariant homeomorphism. First, let $\mathrm{Int}(\Delta[n]) \subset \Delta[n]$

be the interior. The push-out diagram above shows that

$$V = \sigma_y(\mathrm{Int}(\Delta[n])) \subset |Y[-]|$$

is an open subset. Moreover, G acts freely on the set of non-degenerate simplices of $X[-]$ and p identifies the set of orbits with the set of non-degenerate simplices of $Y[-]$. (Prove this.) It follows that, if we choose $x \in X[n]$ with $p(x) = y$, then

$$\varphi : V \times G \to p^{-1}(V),$$

$$\varphi(\sigma_y(a), g) = \sigma_{xg}(a)$$

is a G-equivariant homeomorphism and $p \circ \varphi = pr_1$. So we are done for $v \in \sigma_y(\mathrm{Int}(\Delta[n]))$. It remains to consider

$$v \in |Y[-]| \setminus \sigma_y(\mathrm{Int}(\Delta[n])) = |Z[-]|.$$

By the inductive hypothesis, we can find an open subset $v \in V_1 \subset |Z[-]|$ and a G-equivariant homeomorphism

$$\varphi_1 : V_1 \times G \xrightarrow{\sim} p^{-1}(V_1)$$

such that $p \circ \varphi_1 = pr_1$. Since σ_y is continuous, $\sigma_y^{-1}(V_1) \subset \partial\Delta[n]$ is open, so we can write

$$\sigma_y^{-1}(V_1) = V_2' \cap \partial\Delta[n]$$

with $V_2' \subset \Delta[n]$ open. We choose a particular such $V_2' \subset \Delta[n]$: We let

$$b = \sum_{i \in [n]} \frac{1}{n+1} \cdot i \in \Delta[n]$$

be the barycenter and define the radial projection map

$$\rho : \Delta[n] \setminus \{b\} \longrightarrow \partial\Delta[n]$$

as follows. For $a \in \Delta[n]$, we define $S(a) \subset [n]$ to be the subset

$$i \in S(a) \overset{}{\underset{\text{def}}{\iff}} a_i = \min\{a_j \mid j \in [n]\}$$

Then $S(a) = [n]$ if and only if $a = b$, and we define ρ by

$$\rho(a) = \frac{\sum_{i \in [n] \setminus S(a)} a_i \cdot i}{\sum_{i \in [n] \setminus S(a)} a_i}.$$

Then ρ is continuous, so

$$V_2' = \rho^{-1}(\sigma_\gamma^{-1}(V_1)) \subset \Delta[n]$$

We define $V_2 = \sigma_\gamma(V_2')$ and

$$V = V_1 \cup V_2.$$

Since both $\sigma_\gamma^{-1}(V) = V_2' \subset \Delta[n]$ and $\iota^{-1}(V) = V_1 \subset |Z[-1]|$ are open, the subset $V \subset |Y[-1]|$ is open. We define a G-equivariant homeomorphism

$$\varphi_2 : V_2 \times G \xrightarrow{\sim} p^{-1}(V_2)$$

such that $p \circ \varphi_2 = pr_1$ and such that

$$\varphi_2|_{V_1 \cap V_2} = \varphi_1|_{V_1 \cap V_2}.$$

Then φ_1 and φ_2 together define a G-equivariant homeomorphism

$$\varphi : V \times G \xrightarrow{\sim} p^{-1}(V)$$

with $p \circ \varphi = pr$, as desired. For every $a \in V_2'$ and $g \in G$, there exists a unique non-degenerate simplex $x = x(a,g) \in X[n]$ with $p(x) = g$ and

$$\varphi_1(\sigma_g(\rho(a)), g) = \sigma_{x(a,g)}(\rho(a)).$$

We then define φ_2 by

$$\varphi_2(\sigma_g(a), g) = \sigma_{x(a,g)}(a).$$

This shows that $p\colon |X[-]| \to |Y[-]|$ is a covering space.

Finally, since BG is a CW-complex, it is locally contractible (see Hacther, Alg. Top., Prop. A.4), and hence, admits a universal covering $q\colon \widetilde{BG} \to BG$. Since $\pi_1(EG, e) = 0$, covering space theory shows that there exists a G-equivariant continuous bijection $\tilde{p}$ that makes the diagram

$$\begin{array}{ccc} EG & \xrightarrow{\tilde{p}} & \widetilde{BG} \\ \downarrow p & & \downarrow q \\ BG & = & BG \end{array}$$

commute. Now, in general, a covering space of a CW-complex is again a CW-complex. Therefore, $\tilde{p}$ is a homeomorphism if and only if the restriction of $\tilde{p}$ to the subspaces $|X[-1]| \subset EG$ we considered above are homeomorphism. This is true, since both $p|_{|Y[-1]|}$ and $q|_{|Y[-1]|}$ are covering spaces. //

By definition, the mapping fiber $F(p,*)$ is the limit of the following diagram

$$\begin{tikzcd}[column sep=small]
EG \arrow[dr, "p"] & & \underline{\mathrm{Hom}}([0,1], BG) \arrow[dl, "ev_1"'] \arrow[dr, "ev_0"] & & * \arrow[dl] \\
& BG & & BG &
\end{tikzcd}$$

The map p induces a map to this diagram from the diagram

$$\begin{tikzcd}[column sep=small]
EG \arrow[dr, equal] & & \underline{\mathrm{Hom}}([0,1], EG) \arrow[dl, "ev_1"'] \arrow[dr, "ev_0"] & & p^{-1}(*) \arrow[dl] \\
& EG & & EG &
\end{tikzcd}$$

We write

$$p_* : F(p,*)' \longrightarrow F(p,*)$$

for the induced map of limits. We show that p_* is a homeomorphism. Consider the diagram

$$\begin{tikzcd} F(p,*) \arrow[r, "i"] \arrow[d, "i_0"'] & EG \arrow[d, "p"] \\ F(p,*) \times [0,1] \arrow[r, "ev"] \arrow[ur, dashed, "\widetilde{ev}"] & BG \end{tikzcd}$$

where $ev((e,\sigma),t) = \sigma(t)$. It follows from covering space theory (see Hatcher, Prop. 1.30) that there exists a unique map $\widetilde{ev}$ that make the two triangles commute. By adjunction, we get a map

$$q' : F(p,*) \longrightarrow \underline{\mathrm{Hom}}([0,1], EG).$$

Since $p \circ \widetilde{ev} = ev$, it follows that q' defines a map

$$q : F(p,*) \longrightarrow F(p,*)'$$

inverse to p_*. We have maps

$$j : F(p,*)' \longrightarrow p^{-1}(*)$$

$$j' : p^{-1}(*) \longrightarrow F(p,*)'$$

defined by $j(\tilde{\sigma}) = \tilde{\sigma}(1)$ and $j'(e) = \bar{e}$. Here $\tilde{\sigma} : [0,1] \to EG$ is a path with $\tilde{\sigma}(1) \in p^{-1}(*)$.

$$j \circ j' = id$$

$$j' \circ j \simeq id$$

where the homotopy from the id to $j' \circ j$ is given by

$$h(\tilde{\sigma}, s)(t) = \tilde{\sigma}(s+t-st).$$

Hence, $F(p,*)$ is homotopy equivalent to the discrete space $p^{-1}(*)$, so

$$\pi_n(F(p,*), (e,\bar{*})) = \begin{cases} (p^{-1}(*), e) & (n=0) \\ 0 & (n>0). \end{cases}$$

We the bijection $G \longrightarrow p^{-1}(*)$ given by $g \mapsto g \cdot c$. It depends on the choice of base-point $c \in p^{-1}(*)$. We define the homeomorphism

$$\gamma : (S^1, \infty) \longrightarrow (\Delta[1]/\partial\Delta[1], \overline{\partial\Delta[1]})$$

to be the composition of the inverse of the homeomorphism τ and the homeomorphism of $([-1,1]/\{-1,1\}, \overline{\{-1,1\}})$ onto $(\Delta[1]/\partial\Delta[1], \overline{\partial\Delta[1]})$ defined by

$$t \longmapsto \frac{1-t}{2} \cdot 0 + \frac{1+t}{2} \cdot 1 .$$

The 1-skeleton $sk_1 BG \subset BG$ is:

$$((* \times \Delta[0]) \amalg (G \times \Delta[1]))/\sim$$

$$= (G, 1) \wedge (\Delta[1]/\partial\Delta[1], \overline{\partial\Delta[1]}) .$$

Now, the adjoint of the composition

$$(G, 1) \wedge (S^1, \infty)$$

$$\xrightarrow{id \wedge \gamma} (G, 1) \wedge (\Delta[1]/\partial\Delta[1], \overline{\partial\Delta[1]})$$

$$\hookrightarrow (BG, *)$$

is a map

$$\sigma : (G, 1) \longrightarrow (\Omega(BG, *), \bar{*}).$$

We also define

$$\sigma' : (\Omega(BG, *), \bar{*}) \longrightarrow (G, 1)$$

as follows: let $\widetilde{ev \circ \tau^*}$ be the unique lifting in the diagram

$$\begin{tikzcd} \Omega BG \arrow[r, "\bar{e}"] \arrow[d, "\iota_1"'] & EG \arrow[d, "p"] \\ \Omega BG \times [-1, 1] \arrow[r, "ev \circ \tau^*"] \arrow[ur, dashed, "\widetilde{ev \circ \tau^*}"] & BG \end{tikzcd}$$

We then define $\sigma'(\omega) \in G$ to be the unique element such that

$$(\widetilde{ev \circ \tau^*})(\omega)(1) = \sigma'(\omega) \cdot e.$$

As before, we see that

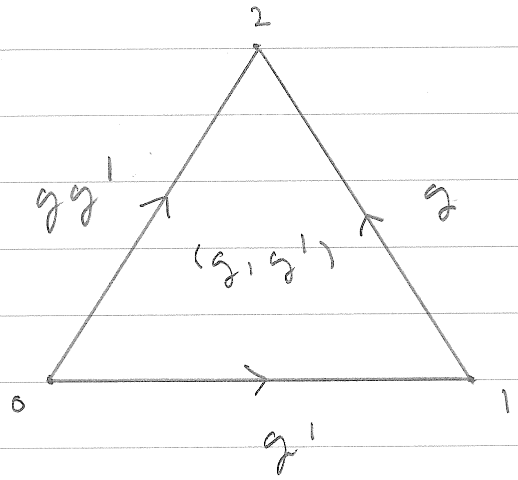
$$\sigma \circ \sigma' = \mathrm{Id}$$

$$\sigma' \circ \sigma \simeq \mathrm{Id}.$$

It follows that σ induces a bijection

$$\sigma_* : G \xrightarrow{\cong} \pi_1(BG, *).$$

We claim that this is an isomorphism of groups. Indeed, by definition, $\sigma_*(g)$, $\sigma_*(g')$, and $\sigma_*(g \cdot g')$ are the homotopy classes of the loops that trace out the 3 1-simplices



in BG. But this triangle is the boundary of the 2-simplex (g, g'). Hence, we have

$$\sigma_*(g \cdot g') = \sigma_*(g) * \sigma_*(g')$$

as desired. The 2-simplex gives the homotopy we need. We have proved that there are canonical isomorphisms (of groups for $n \geq 1$):

$$\pi_n(BG, *) \cong \begin{cases} G & (n=1) \\ 0 & (n \neq 1). \end{cases}$$

So the spaces $(BG, *)$ are examples of spaces (X, x) for which all the homotopy groups $\pi_n(X, x)$ are known. All other examples are infinite loop spaces, i.e. spaces (X, x) for which there exist homotopy equivalences

$$(X, x) \simeq (\Omega(X_1, x_1), \bar{x}_1)$$

$$(X_1, x_1) \simeq (\Omega(X_2, x_2), \bar{x}_2)$$

$$(X_2, x_2) \simeq (\Omega(X_3, x_3), \bar{x}_3)$$

$$\vdots$$

In particular

$$\pi_n(X, x) \cong \pi_{n+k}(X_k, x_k)$$

is an abelian group, for all $k \geq 0$. We can also use this to define $\pi_n(X, x)$, for $n < 0$. But these groups will depend on the spaces (X_k, x_k) and not just the space (X, x).