Pespectives in Mathematical Sciences: Report Problems

Due: Monday, June 28, in Science Building 1, Room 105.

Problem 1. Let D be a division ring, let $R = M_n(D)$, and let $S = M_{n,1}(D)$. We view S as a left R-module and as a right D-vector space.

(i) Let $x \in S$ be a non-zero vector. Show that there exists a matrix $P \in R$ such that $PS = xD \subset S$. (Hint: Try $x = e_1$ first.)

(ii) Let $x, y \in S$ be non-zero vectors. Show that there exists a matrix $A \in R$ such that $Ax = y$.

Problem 2. Let $\varphi(n)$ be Euler's phi-function that counts the number of integers $0 \leqslant k < n$ such that k and n are relatively prime. Show that

$$\sum_{d|n} \varphi(d) = n$$

where the sum ranges over all the divisors d of n.

Problem 3. Let R be a commutative ring and let $\mathfrak{p} \subset R$ be a proper ideal. Show that the following (i)–(ii) are equivalent.

(i) For all elements $a, b \in R$, $ab \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.

(ii) For all ideals $\mathfrak{a}, \mathfrak{b} \subset R$, $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$ implies $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$.

Problem 4. Show that $\mathbb{Z}$ is a Dedekind ring. (Hint: Show that every ideal in $\mathbb{Z}$ is a principal ideal and use Example 4.3.)