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A ring $\longmapsto$ $(X, \mathcal{O}_X)$ ringed space

Point set:

$$X = \mathrm{Spec}(A) = \{\mathfrak{p} \subset A \mid \text{prime ideal}\}$$
$$\begin{array}{ccc} \Psi & & \Psi \\ x & \longleftrightarrow & \mathfrak{p}_x \end{array}$$

$$\mathcal{O}_{X,x} = A_{\mathfrak{p}_x} \quad \text{local ring at } x \in X$$
$$\cup$$
$$\mathfrak{m}_x = \mathfrak{p}_x A_{\mathfrak{p}_x} \quad \text{maximal ideal}$$

$$k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x \quad \text{residue field at } x \in X$$

$$\begin{array}{ccccc} A & \twoheadrightarrow & A/\mathfrak{p}_x & \hookrightarrow & k(x) \\ \Psi & & & & \Psi \\ f & & \longmapsto & & f(x) \end{array} \quad \begin{array}{l} \text{value of } f \\ \text{at } x \in X \end{array}$$

$$\mathfrak{p}_x \ni f \iff f(x) = 0.$$

Topology: Closed sets: $I \subset A$ ideal

$$V(I) = \{x \in X \mid \forall f \in I : f(x) = 0\}$$
$$= \{x \in X \mid \mathfrak{p}_x \supset I\}$$

Satisfy:

(i) $V(\{0\}) = X$, $V(A) = \emptyset$

(ii) $I \subset J \implies V(I) \supset V(J)$

(iii) $\bigcap_{\lambda \in \Lambda} V(I_\lambda) = V(\sum_{\lambda \in \Lambda} I_\lambda)$

(iv) $V(I) \cup V(J) = V(I \cap J) = V(IJ)$

(v) $V(I) = V(\sqrt{I})$,

$$\sqrt{I} = \bigcap_{\mathfrak{p} \supset I} \mathfrak{p} = \{ f \in A \mid f^n \in I, \text{ some } n \geq 1 \}.$$

Pf Prove (iv): By definition,

$$V(I) \cup V(J) \subset V(I \cap J) \subset V(IJ).$$

So suppose $x \notin V(I) \cup V(J)$. Then there exists $f \in I$ and $g \in J$ s.t. $f(x) \neq 0 \neq g(x)$ in $k(x)$. But then

$$0 \neq f(x) \cdot g(x) = (fg)(x) \in k(x),$$

so $x \notin V(IJ)$. Hence, $V(IJ) \subset V(I) \cup V(J)$. //

Distinguished open sets: For $f \in A$,

$$D(f) = \{ x \in X \mid f(x) \neq 0 \}$$

$$= \{ x \in X \mid \mathfrak{p}_x \not\ni f \}$$

Note that $D(f) \cap D(g) = D(fg)$.

Form a basis of the open sets:

$$X \setminus V(I) = \{ x \in X \mid \exists f \in I : f(x) \neq 0 \}$$

$$= \bigcup_{f \in I} D(f).$$

<u>Lemma</u> The following are equivalent

(i) $\operatorname{Spec}(A) = \bigcup_{\lambda \in \Lambda} D(f_\lambda)$

(ii) $A = (f_\lambda \mid \lambda \in \Lambda)$.

<u>Pf</u> (i) $\Leftrightarrow$ $\forall x \in X, \exists \lambda \in \Lambda : f_\lambda(x) \neq 0$ $\Leftrightarrow$

$\forall x \in X, \exists \lambda \in \Lambda : f_\lambda \notin \mathfrak{p}_x$ $\Leftrightarrow$

$\forall x \in X : (f_\lambda \mid \lambda \in \Lambda) \not\subseteq \mathfrak{p}_x$ $\Leftrightarrow$ (ii).

Here the last "$\Rightarrow$" uses that every ideal is contained in some maximal ideal. //

<u>Cor</u> $\operatorname{Spec}(A)$ is quasi-compact.

<u>Pf</u> Suppose $\operatorname{Spec}(A) = \bigcup_{\lambda \in \Lambda} U_\lambda$; can assume $U_\lambda = D(f_\lambda)$. By lemma, $1 \in (f_\lambda \mid \lambda \in \Lambda)$; say $1 = f_{\lambda_1} g_1 + \dots + f_{\lambda_n} g_n$. But then $(f_{\lambda_1}, \dots, f_{\lambda_n}) = A$, so

$$\operatorname{Spec}(A) = U_{\lambda_1} \cup \dots \cup U_{\lambda_n}. \qquad //$$

Structure sheaf $\mathcal{O}_X$: Write

$$A_f = A[\tfrac{1}{f}] = \{1, f, f^2, \dots\}^{-1} A.$$

Lemma $D(f) \supset D(g) \iff g \in \sqrt{(f)}$.

Pf $D(f) \not\supset D(g) \iff \exists x \in X : x \notin D(f)$ and $x \in D(g)$

$\iff \exists x \in X : f \in \mathfrak{p}_x$ and $g \notin \mathfrak{p}_x \iff$

$$g \notin \bigcap_{\mathfrak{p} \ni f} \mathfrak{p} = \sqrt{(f)}. \quad //$$

If $D(f) \supset D(g)$, then the canonical map

$$A_f \xrightarrow{\rho_{g,f}} A_g$$

is the ring homomorphism defined by

$$\rho_{g,f}(a/f^i) = ab^i/g^{ni}$$

where $g = bf^n$. It is well-defined, and if $D(f) \supset D(g) \supset D(g)$, then the diagram of canonical maps

$$\begin{tikzcd} A_f \arrow[rr, "\rho_{g,f}"] \arrow[dr, "\rho_{h,f}"'] & & A_g \arrow[dl, "\rho_{h,g}"] \\ & A_h & \end{tikzcd}$$

commutes. In particular, if $D(f) = D(g)$, then the canonical maps

$$A_f \underset{\rho_{f,g}}{\overset{\rho_{g,f}}{\rightleftarrows}} A_g$$

are each others inverses. In general, however, $A_f \neq A_g$. Also, if $x \in D(f)$, then $f \in A \setminus \mathfrak{p}_x$, so there is a can. ring homomorphism

$$A_f \xrightarrow{\rho_x^f} A_{\mathfrak{p}_x}$$

defined by $\rho_x^f(a/f^i) = a/f^i$.

Prop If $D(f) = \bigcup_{\alpha \in \mathcal{A}} D(f_\alpha)$, then the following diagram is an equalizer.

$$A_f \xrightarrow{i} \prod_{\alpha \in \mathcal{A}} A_{f_\alpha} \overset{j}{\underset{k}{\rightrightarrows}} \prod_{\alpha,\beta \in \mathcal{A}} A_{f_\alpha f_\beta}$$

Here i, j, and k are defined by:

$$pr_\alpha \circ i = \rho_{f_\alpha, f}$$

$$pr_{\alpha,\beta} \circ j = \rho_{f_\alpha f_\beta, f_\alpha} \circ pr_\alpha$$

$$pr_{\alpha,\beta} \circ k = \rho_{f_\alpha f_\beta, f_\beta} \circ pr_\beta .$$

Pf Show that i is injective. Let $g \in A_f$ and suppose $\rho_{f_\alpha, f}(g) = 0 \in A_{f_\alpha}$ for all $\alpha \in \mathfrak{A}$. Write $g = b/f^n$ and let

$$I = \mathrm{Ann}_A(b) = \{ c \in A \mid bc = 0 \}$$

be the annihilator ideal. The following are equivalent:

(i) $g = 0$ in A_f

(ii) $\exists\, m \geq 0 : f^m b = 0 \in A$.

(iii) $f \in \sqrt{I}$

(iv) $\mathfrak{p}_x \supset I \Longrightarrow f \in \mathfrak{p}_x$

(v) $\forall x \in V(I) : f(x) = 0$.

We wish to show that $g = 0 \in A_f$. If not, then there exists $x \in V(I)$ s.t. $f(x) \neq 0$. So $x \in D(f)$, and hence, $x \in D(f_\alpha)$, for some $\alpha \in \mathfrak{A}$. By assumption,

$$\begin{tikzcd} A_f \arrow[r, "\rho_{f_\alpha, f}"] \arrow[rr, bend right, "\rho^f_x"'] & A_{f_\alpha} \arrow[r, "\rho^{f_\alpha}_x"] & A_{\mathfrak{p}_x} \end{tikzcd}$$

$$g \longmapsto 0 \longmapsto 0$$

and hence, $\rho_x^f(b) = \rho_x^f(gf^n) = 0 \in A_{\mathfrak{p}_x}$. But then there exists $c \in A \setminus \mathfrak{p}_x$ s.t. $bc = 0 \in A$, i.e. $c \in I$. But $x \in V(I)$, so $\mathfrak{p}_x \supset I$, and hence, $c \in \mathfrak{p}_x$, a contradiction. So $g = 0$. This proves that i is injective.

Next, let $g_\alpha \in A_{f_\alpha}$, $\alpha \in \mathcal{A}$, and suppose that for all $\alpha, \beta \in \mathcal{A}$,

$$\rho_{f_\alpha f_\beta, f_\alpha}(g_\alpha) = \rho_{f_\alpha f_\beta, f_\beta}(g_\beta)$$

in $A_{f_\alpha f_\beta}$. We must show that there exists $g \in A_f$ with

$$\rho_{f_\alpha, f}(g) = g_\alpha$$

for all $\alpha \in \mathcal{A}$. We only consider the case $f = 1$; the general case is similar. By compactness, we can choose $f_{\alpha_1}, \dots, f_{\alpha_r} \in \{ f_\alpha \mid \alpha \in \mathcal{A} \}$ s.t.

$$A = (f_{\alpha_1}, \dots, f_{\alpha_r}).$$

It will suffice to find $g \in A_f$ s.t.

$$\rho_{f_{\alpha_i}, f}(g) = g_{\alpha_i}$$

for $i = 1, \dots, r$. Indeed, in this case, we have for every $\alpha \in \mathcal{A}$ that

$$\rho_{f_\alpha f_{\alpha_i}, f_\alpha}(\rho_{f_\alpha, f}(g))$$

$$= \rho_{f_\alpha f_{\alpha_i}, f_{\alpha_i}}(\rho_{f_{\alpha_i}, f}(g))$$

$$= \rho_{f_\alpha f_{\alpha_i}, f_{\alpha_i}}(g_{\alpha_i})$$

$$= \rho_{f_\alpha f_{\alpha_i}, f_\alpha}(g_\alpha)$$

for all $i = 1, \ldots, r$. By what was proved first, we conclude that

$$\rho_{f_\alpha, f}(g) = g_\alpha$$

as desired.

To find $g \in A_f$, we write

$$g_{\alpha_i} = b_i / f_{\alpha_i}^n \in A_{f_{\alpha_i}}$$

with n independent of i. We have

$$\begin{array}{ccc} \dfrac{b_i f_{\alpha_j}^n}{(f_{\alpha_i} f_{\alpha_j})^n} & = & \rho_{f_{\alpha_i} f_{\alpha_j}, f_{\alpha_i}}(g_{\alpha_i}) \\ & & \parallel \\ \dfrac{b_j f_{\alpha_i}^n}{(f_{\alpha_i} f_{\alpha_j})^n} & = & \rho_{f_{\alpha_i} f_{\alpha_j}, f_{\alpha_j}}(g_{\alpha_j}) \end{array}$$

and hence there exists $m \geq 0$ s.t.

$$(f_{\alpha_i} f_{\alpha_j})^m \left(b_i f_{\alpha_j}^n - b_j f_{\alpha_i}^n \right) = 0 \in A$$

with m independent of i and j.
We let $p = m+n$ and $b_i' = b_i f_{\alpha_i}^m$ s.t.

$$g_{\alpha_i} = b_i' / f_{\alpha_i}^p \in A_{f_{\alpha_i}}$$

$$b_i' f_{\alpha_j}^p = b_j' f_{\alpha_i}^p \in A.$$

Since

$$X = \bigcup_{i=1}^r D(f_{\alpha_i}) = \bigcup_{i=1}^r D(f_{\alpha_i}^p),$$

we have $1 \in (f_{\alpha_1}^p, \ldots, f_{\alpha_r}^p)$, say

$$1 = f_{\alpha_1}^p h_1 + \cdots + f_{\alpha_r}^p h_r.$$

Now, let

$$g = b_1' h_1 + \cdots + b_r' h_r.$$

We have

$$\begin{aligned} f_{\alpha_j}^p g &= \sum_{i=1}^r f_{\alpha_j}^p b_i' h_i \\ &= \sum_{i=1}^r f_{\alpha_i}^p b_j' h_i = b_j' \end{aligned}$$

which shows that

$$\rho_{f\alpha_j, f}(g) = b'_j / f^p_{\alpha_j} = g_{\alpha_j}$$

as desired. //

We now define the sheaf $\mathcal{O}_X$, up to unique isomorphism, as follows. If $U \subset X$ is an open set, we let $D(U)$ be the category with

$$\text{ob}\, D(U) = \{ D(f) \subset U \}$$

and with a unique morphism from $D(g)$ to $D(f)$ if $D(f) \supset D(g)$. There is a functor

$$D(U)^{op} \xrightarrow{F_U} \text{Rings}$$

defined, up to unique isom., by

$$\begin{array}{ccc} F_U(D(f)) & = & A_f \\ \downarrow & & \downarrow \rho_{g,f} \\ F_U(D(g)) & = & A_g \end{array}$$

We define

$$\Gamma(U, \mathcal{O}_X) = \lim_{D(U)^{op}} F_U$$

which we also write simply as

$$\Gamma(U, \mathcal{O}_X) = \lim_{D(f) \subset U} A_f .$$

If $U \subset V \subset X$ are both open, then the inclusion $U \subset V$ gives rise to an inclusion functor $i_U^V : D(U) \to D(V)$. Moreover, there is a unique isom.

$$F_V \circ i_U^V \xrightarrow{u_U^V} F_U$$

and we define

$$\begin{array}{ccc}
\Gamma(V, \mathcal{O}_X) & = & \lim\limits_{D(V)^{op}} F_V \\
\Big\downarrow & & \Big\downarrow \text{res} \\
\rho_U^V & & \lim\limits_{D(U)^{op}} F_V \circ i_U^V \\
\Big\downarrow & & \Big\downarrow u_U^V \\
\Gamma(U, \mathcal{O}_X) & = & \lim\limits_{D(U)^{op}} F_U
\end{array}$$

If $U \subset V \subset W \subset X$, then

$$\rho_U^V \circ \rho_V^W = \rho_U^W$$

and also $\rho_U^U = \mathrm{id}_{\Gamma(U, \mathcal{O}_X)}$. So this defines a presheaf $\mathcal{O}_X$ on X.

If $U = D(f)$, then $U \in \text{ob } D(U)$ is a terminal object, and hence, the canonical map

$$A_f \longrightarrow \Gamma(D(f), \mathcal{O}_X)$$

is an isomorphism. Moreover, if $D(f) \supset D(g)$, then the diagram of canonical maps

$$\begin{array}{ccc} A_f & \longrightarrow & \Gamma(D(f), \mathcal{O}_X) \\ \downarrow \rho_{g,f} & & \downarrow \rho^{D(f)}_{D(g)} \\ A_g & \longrightarrow & \Gamma(D(g), \mathcal{O}_X) \end{array}$$

commutes.

<u>Prop</u> The presheaf $\mathcal{O}_X$ is a sheaf.

<u>Pf</u> We must show that for every open $U \subset X$ and every covering $\{U_\alpha\}$ of U, the following diagram is an equalizer.

$$\Gamma(U, \mathcal{O}_X) \xrightarrow{i} \prod_\alpha \Gamma(U_\alpha, \mathcal{O}_X) \underset{k}{\overset{j}{\rightrightarrows}} \prod_{\alpha,\beta} \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X).$$

We know from the prop. on p. 5 that this is true if U and the U_α are distignished open. The general case follows from the definition of limit. See EGA I (3.2.1) for details. //

Note in particular that

$$\Gamma(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}) \xleftarrow{\sim} A.$$