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An *infinitesimal thickening* is a closed immersion

$$Y \xrightarrow{i} \tilde{Y}$$

s.t. the quasicoherent ideal $I \subset \mathcal{O}_{\tilde{Y}}$ that defines i is nilpotent. In this case, the underlying map of spaces is a homeomorphism. For i is a homeomorphism onto a closed subspace of $\tilde{Y}$, so we just need to see that i is surjective. We may assume $Y = \mathrm{Spec}(B)$ and $\tilde{Y} = \mathrm{Spec}(\tilde{B})$ are affine, $B = \tilde{B}/I$. As spaces,

$$\mathrm{Spec}(B) \underset{i}{\overset{\sim}{\hookrightarrow}} V(I) \subset \mathrm{Spec}(\tilde{B}),$$

and if $I^N = 0$, then

$$V(I) = V(I^N) = V(0) = \mathrm{Spec}(B).$$

Def A morphism of schemes

$$X \xrightarrow{f} S$$

is *formally unramified* (resp. *formally smooth*, resp. *formally étale*) if, in every commutative diagram of schemes of the form

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, hook, "i"] & X \arrow[d, "f"] \\ \tilde{Y} \arrow[r, "h"] & S \end{tikzcd}$$

with i an infinitesimal thickening of affine schemes, there exists at least one (resp. at most one, resp. precisely one) morphism

$$\tilde{Y} \xrightarrow{k} X$$

s.t. $k \circ i = g$ and $f \circ k = h$. "

<u>Rmk</u> It suffices to consider i s.t. the quasicoherent ideal $I \subset \mathcal{O}_{\tilde{Y}}$ has $I^2 = 0$. Also, if X and S are both affine, then a lift in the diagram at top of the page exist if and only one does in

$$\begin{tikzcd} \Gamma(Y, \mathcal{O}_Y) & \Gamma(X, \mathcal{O}_X) \arrow[l] \\ \Gamma(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) \arrow[u, two heads] & \Gamma(S, \mathcal{O}_S) \arrow[l] \arrow[u] \end{tikzcd}$$

"

Lemma An open immersion is formally étale, and a closed immersion is formally unramified.

Pf Let $j : U \hookrightarrow X$ be an open immersion, and consider

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, hook, "i"'] & U \arrow[d, hook, "j"] \\ \tilde{Y} \arrow[r, "h"] \arrow[ur, dashed, "k"] & X \end{tikzcd}$$

with i an infinitesimal thickening. In the underlying diagram of sets (in fact spaces), there exists a unique map k making the two triangles commute. But

$$\begin{tikzcd} \mathrm{Hom}_{Sch}(\tilde{Y}, U) \arrow[r, "j_*"] \arrow[d, "\text{forget}"] & \mathrm{Hom}_{Sch}(\tilde{Y}, X) \arrow[d, "\text{forget}"] \\ \mathrm{Hom}_{Set}(\tilde{Y}, U) \arrow[r, "j_*"] & \mathrm{Hom}_{Set}(\tilde{Y}, X) \end{tikzcd}$$

is cartesian, by the definition of open immersion. Hence, there exists a unique map of schemes making the two triangles commute.

Next, let $i' : Z \hookrightarrow X$ be a closed immersion and consider a commutative diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d, hook, "i'"'] & Z \arrow[d, hook, "i'"] \arrow[dl, shift right, "f"'] \arrow[dl, shift left, "g"] \\ \tilde{Y} \arrow[r] & X \end{tikzcd}$$

with i an infinitesimal thickening. Since i induces a bijection of the underlying sets, we have $f = g$ as maps of spaces, by commutativity of the top triangles. Since i' is a closed immersion, $i'^{\#}: \mathcal{O}_X \to i'_* \mathcal{O}_Z$ is surjective, and since i'^* is exact, and $\varepsilon: i'^* i'_* \mathcal{O}_Z \xrightarrow{\sim} \mathcal{O}_Z$, so is the left-hand map in

$$i'^* \mathcal{O}_X \longrightarrow \mathcal{O}_Z \overset{f^{\#}}{\underset{g^{\#}}{\rightrightarrows}} f_* \mathcal{O}_{\tilde{Y}} = g_* \mathcal{O}_{\tilde{Y}} .$$

But the two composites agree, by commutativity of the lower triangles, so $f = g$ as morphisms of schemes. //

We proceed to express the notions introduced on page 1-2 in terms of sheaves of differentials, and begin by proving the following general result.

Prop If $X \xrightarrow{f} S \xrightarrow{g} T$ are morphisms of schemes, then the induced sequence of (quasicoherent) $\mathcal{O}_X$-modules

$$\mathcal{O}_X \underset{f^*\mathcal{O}_S}{\otimes} f^*\Omega^1_{S/T} \longrightarrow \Omega^1_{X/T} \longrightarrow \Omega^1_{X/S} \longrightarrow 0$$

is exact.

Pf The (large) category $\mathrm{Mod}_{\mathcal{O}_X}$ of $\mathcal{O}_X$-modules is abelian and, as is true for every abelian category, the (additive) Yoneda embedding

$$\mathrm{Mod}_{\mathcal{O}_X}^{op} \longrightarrow (\text{add. functors } \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathbb{Z}})$$

$$M \longmapsto \mathrm{Hom}_{\mathcal{O}_X}(M, -)$$

reflects exactness. This means that a sequence of $\mathcal{O}_X$-modules

$$M' \longrightarrow M \longrightarrow M''$$

is exact if, for every $\mathcal{O}_X$-module N, the induced sequence of abelian groups

$$\mathrm{Hom}_{\mathcal{O}_X}(M', N) \leftarrow \mathrm{Hom}_{\mathcal{O}_X}(M, N) \leftarrow \mathrm{Hom}_{\mathcal{O}_X}(M'', N)$$

is exact. In the case at hand, if

N is an $\mathcal{O}_X$-module, then

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X \otimes_{f^*\mathcal{O}_S} \Omega^1_{S/T}, N) \longleftarrow \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/T}, N)$$
$$\longleftarrow \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/S}, N) \longleftarrow 0$$

is canonically isomorphic to

$$\mathrm{Der}_T(\mathcal{O}_S, f_*N) \longleftarrow \mathrm{Der}_T(\mathcal{O}_X, N)$$
$$\longleftarrow \mathrm{Der}_S(\mathcal{O}_X, N) \longleftarrow 0$$

which is exact by definition. //

Prop If $f : X \longrightarrow S$ is formally smooth and $g : S \longrightarrow T$ is any morphism, then

$$0 \longrightarrow \mathcal{O}_X \otimes_{f^*\mathcal{O}_S} \Omega^1_{S/T} \longrightarrow \Omega^1_{X/T} \longrightarrow \Omega^1_{X/S} \longrightarrow 0$$

is exact.

Pf Since exactness can be determined locally, we may assume that X, S, and T are affine. In this case, we prove that there exists a retraction

$$\mathcal{O}_X \otimes_{f^*\mathcal{O}_S} \Omega^1_{S/T} \xleftarrow{\ r^\# \ } \Omega^1_{X/T}$$

of the left-hand map. Since f is formally smooth, the lifts $\tilde{d}_0$ and $\tilde{d}_1$ in the following diagram exist.

$$\begin{tikzcd}
X \arrow[rr, equal] \arrow[d, hook, "i"'] & & X \arrow[d] \arrow[r, equal] & X \arrow[d] \\
X \times_S P^1_{S/T} \arrow[r, "pr_2"] \arrow[urr, shift left, "\tilde{d}_0"] \arrow[urr, shift right, "\tilde{d}_1"'] & P^1_{S/T} \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & S \arrow[r, "g"] & T
\end{tikzcd}$$

Indeed, the morphism ε is an infinitesimal thickening of affine schemes defined by the square-zero quasicoherent ideal

$$i_*(\mathcal{O}_X \otimes_{f^*\mathcal{O}_S} \Omega^1_{S/T}) \subset i_*(\mathcal{O}_X \otimes_{f^*\mathcal{O}_S} P^1_{S/T}).$$

Moreover, we have $g \circ d_0 = g \circ d_1$, so we obtain the following diagram

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, hook] & X \arrow[r, equal] \arrow[d, hook] & X \arrow[d, "\Delta_{X/T}"] \\
X \times_S P^1_{S/T} \arrow[r, "r"] \arrow[rr, bend right, "(\tilde{d}_0, \tilde{d}_1)"'] & P^1_{X/T} \arrow[r, hook] & X \times_T X
\end{tikzcd}$$

The morphism r determines and is determined by a map of $\mathcal{O}_X$-modules

$$\mathcal{O}_X \underset{f^*\mathcal{O}_S}{\otimes} \Omega^1_{S/T} \xleftarrow{r^\#} \Omega^1_{X/T}$$

By construction, the diagram

$$\begin{tikzcd} X \times_S P^1_{S/T} \arrow[r, "r"] \arrow[rr, bend right, "pr_2"'] & P^1_{X/T} \arrow[r, "f"] & P^1_{S/T} \end{tikzcd}$$

commutes, and hence, so does

$$\begin{tikzcd} \mathcal{O}_X \underset{f^*\mathcal{O}_S}{\otimes} \Omega^1_{S/T} & \Omega^1_{X/T} \arrow[l, "r^\#"'] & \Omega^1_{S/T} \arrow[l, "f^*"'] \arrow[ll, bend left, "in_2"] \end{tikzcd}$$

This completes the proof. //

<u>Lemma</u> Let $f: X \to S$ be a formally unramified (resp. étale) morphism. Then, in every commutative diagram

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ \tilde{Y} \arrow[r, "h"] & S \end{tikzcd}$$

with i an infinitesimal thickening, at most one (resp. exactly one) morphism $k : \tilde{Y} \to X$ such that $k \circ i = g$ and $f \circ k = h$ exists.

Pf Suppose first that $f : X \to S$ is formally unramified. By definition, the restriction of a morphism $k : \tilde{Y} \to X$ with $k \circ i = g$ and $f \circ k = h$ to every affine open subscheme $\tilde{U} \subset \tilde{Y}$ is unique; hence k is unique. Similarly, if $f : X \to S$ is formally étale, then for all affine open subschemes $\tilde{U} \subset \tilde{Y}$ and $U = \tilde{U} \times_{\tilde{Y}} Y \subset Y$, a unique lift k_U exists in

$$\begin{tikzcd} U \arrow[r, "g|_U"] \arrow[d, "i|_U"'] & X \arrow[d, "f"] \\ \tilde{U} \arrow[r, "h|_U"'] \arrow[ur, "k_U"] & S \end{tikzcd}.$$

By uniqueness, these morphism glue to give the desired morphism $k : \tilde{Y} \to X$. //

Rmk For formally smooth morphisms, the analogous statement is false. //

Prop Let $f : X \to S$ be a morphism. The following are equivalent.

(1) f is formally unramified.

(2) $\Omega^1_{X/S} = 0$.

Pf If (1) holds, then the lemma shows that, in the diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "i"'] & X \arrow[d, "f"] \\ P^1_{X/S} \arrow[r] \arrow[ur, shift left, "d_0"] \arrow[ur, shift right, "d_1"'] & S \end{tikzcd},$$

we have $d_0 = d_1 : P^1_{X/S} \to X$. Hence,

$$d = 0 : \mathcal{O}_X \to \Omega^1_{X/S},$$

and since the image of d locally generates $\Omega^1_{X/S}$ as an $\mathcal{O}_X$-module, this module is zero. So (2) holds.

Conversely, if $\Omega^1_{X/S} = 0$, then the closed immersion $i' : X \to P^1_{X/S}$ is an isomorphism. Now, suppose that we are given two lifts k_0, k_1 in a diagram

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, hook, "i'"'] & X \arrow[d] \\ \tilde{Y} \arrow[r, "h"] \arrow[ur, shift left, "k_0"] \arrow[ur, shift right, "k_1"'] & S \end{tikzcd}$$

where i' is an infinitesimal thickening defined by a quasicoherent ideal $I \subset \mathcal{O}_{\tilde{Y}}$ with $I^2 = 0$. We obtain a commutative diagr.

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, hook, "i'"'] & X \arrow[r, equal] \arrow[d, hook, "\tilde{i}'"] & X \arrow[d, "\Delta_{X/S}"] \\ \tilde{Y} \arrow[r, "k"] \arrow[rr, bend right, "{(k_0, k_1)}"'] & P^1_{X/S} \arrow[r, "{(d_0, d_1)}"] & X \times_S X \end{tikzcd}$$

and since i' is an isomorphism, we conclude that $k_0 = k_1$. So (1) holds.

<u>Cor</u> If $f: X \to S$ is formally étale, then for every morphism $g: S \to T$, the canonical map of $\mathcal{O}_X$-modules

$$\mathcal{O}_X \otimes_{f^*\mathcal{O}_S} \Omega^1_{S/T} \longrightarrow \Omega^1_{X/S}$$

is an isomorphism. //