 2013.06.10

We introduce a variety of finiteness notions, all of which are relative and preserved under composition and base-change. By contrast the notion of being noetherian is absolute, and therefore, is not a good basic notion.

Def A morphism of schemes $f: X \to S$ is quasi-compact if for every quasi-compact open $V \subset S$, $f^{-1}(V) \subset X$ is quasi-compact. //

Lemma Let $f: X \to S$ be a morphism. The following are equivalent.

(1) f is quasi-compact.

(2) For every $V \subset S$ affine open, $f^{-1}(V) \subset X$ is quasi-compact.

(3) There exists a covering $\{V_i\}_{i \in I}$ of S by affine open subsets s.t. for all $i \in I$, $f^{-1}(V_i) \subset X$ is quasi-cpt.

Pf (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is trivial. So we must show (3) $\Rightarrow$ (1). Let $\{V_i\}_{i \in I}$ be as in (3) and let $V \subset S$ be affine open. Let $s \in V$ and choose

$i = i(s) \in I$ s.t. $s \in V_i$. Since V and V_i are both affine, we can find $s \in W_s \subset V \cap V_i$ s.t. W_s is a distinguished open in both V and V_i. Now, since V is quasi-compact,

$$V = W_{s_1} \cup \cdots \cup W_{s_n}$$

for $s_1, \dots, s_n \in V$, a finite number. For each $j = 1, \dots, n$, we write

$$f^{-1}(V_{i(s_j)}) = \bigcup_{1 \le k \le l_j} U_{jk}$$

as a finite union of affine open subsets, using (3). Since $W_{s_j} \subset V_{i(s_j)}$ is a distinguished open, so is

$$f^{-1}(W_{s_j}) \cap U_{jk} \subset U_{jk}$$

for all $k = 1, \dots, l_j$. Hence, $f^{-1}(W_{s_j})$ is a finite union of affine open subsets, hence quasi-compact. It follows that $f^{-1}(V)$ is quasi-cpt., so (2) holds. Finally, if $V \subset S$ is any quasi-compact open, then we can write $V = V_1 \cup \cdots \cup V_n$ with $V_1, \dots, V_n$ affine open. So

$$f^{-1}(V) = f^{-1}(V_1) \cup \cdots \cup f^{-1}(V_n)$$

is a finite union of quasi-compact open subsets, hence quasi-compact. //

It is clear from the definition that the composition of two quasi-compact morphisms is quasi-compact.

<u>Lemma</u> Let $f: X \to S$ be a quasi-compact morphism, let $g: S' \to S$ be any morphism, and let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

be a pullback diagram. Then the morphism $f': X' \to S'$ is quasi-compact.

<u>Pf</u> Let $s' \in S'$, let $g(s') \in T \subset S$ be an open affine neighborhood, and let $s' \in W \subset g^{-1}(T)$ be an open affine neighborhood. By the lemma on page 1, it will suffice to show that $f'^{-1}(W) \subset X'$ is quasi-compact. Hence, it will suffice to show that if S and S' are both affine, then $X' = X \times_S S'$ is

quasi-compact. But $X = f^{-1}(S)$ is quasi-compact, and hence, a union of finitely many affine open subsets $U_1, \dots, U_n$. From our construction of $X' = X \times_S S'$ in lecture 6, it follows that X' is the union of the finitely many affine open subsets

$$U_i' = U_i \times_S S' \quad , \quad i = 1, \dots, n.$$

So X' is quasi-compact. //

Ex 1) Every morphism $f: X \to S$ with X and S affine is quasi-compact, by the lemma on page 2.

2) A closed immersion is quasi-cpt.

3) The fold map

$$X = \coprod_{i \in I} S \xrightarrow{\nabla} S$$

is quasi-compact if and only if I is finite. //

Def A morphism of schemes $f: X \to S$ is quasi-separated if the diagonal map $\Delta_{X/S}: X \to X \times_S X$ is quasi-compact.

<u>Lemma</u> Quasi-separated morphisms are preserved under composition and under base-change.

<u>Pf</u> Suppose $f: X \to S$ and $g: S \to T$ are quasi-separated. The diagram

$$\begin{tikzcd} X \arrow[r, "\Delta_{X/S}"] \arrow[d, equal] & X \times_S X \arrow[r] \arrow[d, "g'"] \arrow[dr, phantom, "\text{cart.}"] & S \arrow[d, "\Delta_{S/T}"] \\ X \arrow[r, "\Delta_{X/T}"] & X \times_T X \arrow[r] & S \times_T S \end{tikzcd}$$

and the lemma on page 3 shows that $\Delta_{X/T}$ is quasi-compact, being the composite of the quasi-compact morphisms $\Delta_{X/S}$ and g'. So $g \circ f$ is quasi-separated. Similarly,

$$\begin{tikzcd} X \times_S S' \arrow[r, equal] \arrow[d, "\Delta_{X \times_S S'/S'}"] & X \times_S S' \arrow[d, "\Delta_{X/S} \times_S S'"] \\ (X \times_S S') \times_{S'} (X \times_S S') \arrow[r, "\sim"] & (X \times_S X) \times_S S' \end{tikzcd}$$

shows that a base-change of a quasi-separated morphism is quasi-separated. //

Ex 1) A separated morphism is quasi-separated. In particular, every immersion is quasi-separated.

2) Every morphism $f: X \to S$ with X and S affine is quasi-separated. //

A scheme X is defined to be <u>quasi-separated</u> if the morphism

$$X \longrightarrow \operatorname{Spec}(\mathbb{Z})$$

is quasi-separated. In this case, the intersection of two affine open subsets $U, V \subset X$ is quasi-compact, hence a union of finitely many affine open subsets. Indeed,

$$U \cap V = \Delta_X^{-1}(U \times V),$$

and $U \times V \subset X \times X$ is affine. (If X is separated, then $U \cap V \subset X$ is again affine.) The converse statement also holds; for a proof, see EGA $\overline{\text{IV}}$ 1.2.7.

<u>Rmk</u> If $f: X \to S$ is quasi-compact and quasi-separated and if M is a quasi-coherent $\mathcal{O}_X$-module, then $f_* M$

is a quasi-coherent $\mathcal{O}_S$-module. Similarly, the higher derived image sheaves $R^q f_*(M)$, calculated by using an injective resolution in the category of all $\mathcal{O}_X$-modules, are quasi-coherent $\mathcal{O}_S$-modules; see EGA III 1.4.10, IV 1.7.21. //

Let $f: A \to B$ be a ring homomorphism. We will say that f is of finite type if there exists finitely many elements $b_1, \ldots, b_n \in B$ s.t. the unique ring homomorphism

$$A[x_1, \ldots, x_n] \xrightarrow{g} B$$

that extends f and maps x_i to b_i is surjective. If we can find $b_1, \ldots, b_n \in B$ s.t., in addition, the kernel of g is a finitely generated ideal of $A[x_1, \ldots, x_n]$, then we say that f is of finite presentation. (If A is noetherian, then every ring homomorphism $f: A \to B$ of finite type is automatically of finite presentation.)

<u>Def</u> A morphism of schemes $f: X \to S$ is <u>locally</u> <u>of</u> <u>finite</u> <u>type</u> (resp. <u>locally</u> <u>of</u> <u>finite</u> <u>presentation</u>) if S admits a covering $\{V_i\}_{i \in I}$ by affine open subsets s.t. for all $i \in I$, $f^{-1}(V_i)$ admits a covering $\{U_{ij}\}_{j \in J_i}$ by affine open subsets with the property that the ring homomorphism

$$\Gamma(V_i, \mathcal{O}_S) \longrightarrow \Gamma(U_{ij}, \mathcal{O}_X)$$

induced by f is of finite type (resp. of finite presentation). //

<u>Rmk</u> The notions are local on X. For example, the fold map

$$X = \coprod_{j \in J} S \xrightarrow{\nabla} S$$

is locally of finite presentation for all J. //

<u>Lemma</u> Morphisms locally of finite type (resp. locally of finite presentation) are preserved under composition and under base-change.

<u>Pf</u> Exercise. //

<u>Ex</u> 1) Every (local) immersion is locally of finite type.

2) Every local isomorphism is locally of finite presentation.

<u>Prop</u> A ring homomorphism $f: A \to B$ is of finite type (resp. of finite presentation) if and only if the induced morphism of schemes

$$\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$$

is locally of finite type (resp. locally of finite presentation).

<u>Pf</u> The "only if" part is trivial. The "if" part is proved in EGA I 6.3.3 and EGA $\underline{\mathrm{IV}}$ 1.4.6, respectively. //

<u>Lemma</u> If $f: X \to S$ is locally of finite type, then the diagonal

$$X \xrightarrow{\Delta_{X/S}} X \times_S X$$

is locally of finite presentation.

<u>Pf</u> Let $x \in X$ and choose affine open neighborhoods $x \in U \subset X$ and $f(x) \in V \subset S$

s.t. $\Gamma(V, \mathcal{O}_S) \to \Gamma(U, \mathcal{O}_X)$ is of finite type. Then $\Delta_{X/S}(x) \in U \times_V U \subset X \times_S X$ is an affine open neighborhood and

$$\begin{array}{ccc} \Gamma(U \times_V U, \mathcal{O}_{X \times_S X}) & \longrightarrow & \Gamma(U, \mathcal{O}_X) \\ \| & & \| \\ \Gamma(U, \mathcal{O}_X) \otimes_{\Gamma(V, \mathcal{O}_S)} \Gamma(U, \mathcal{O}_X) & \xrightarrow{\mu} & \Gamma(U, \mathcal{O}_X) \end{array}$$

is of finite presentation. Indeed, if $a_1, \dots, a_n$ generate $\Gamma(U, \mathcal{O}_X)$ as a $\Gamma(V, \mathcal{O}_S)$-algebra, then the elements $1 \otimes a_i - a_i \otimes 1$, $i = 1, \dots, n$, generate the kernel of μ. //

Def A morphism of schemes $f : X \to S$ is of finite type if f is quasi-compact and locally of finite type. //

Prop A morphism $f : X \to S$ is of finite type if and only if for every affine open $V \subset S$, $f^{-1}(V) \subset X$ is a finite union of affine opens $U_1, \dots, U_n \subset X$ with the property that the ring homomorphism

$$\Gamma(V, \mathcal{O}_S) \longrightarrow \Gamma(U_i, \mathcal{O}_X)$$

induced by f is of finite type.

Pf See EGA I 6.3.2. //

The following notion is very important and we will return to it in future lectures, time permitting.

Def A morphism of schemes $f: X \to S$ is proper if it is separated, of finite type, and universally closed. //

Ex 1) Closed immersions are proper.

2) The canonical projection

$$\begin{array}{ccc} \mathrm{Spec}(k[x,y]) & \xrightarrow{f} & \mathrm{Spec}(k[x]) \\ \| & & \| \\ \mathbb{A}^2_k & \xrightarrow{f} & \mathbb{A}^1_k \end{array}$$

is not closed, hence not proper. For $V(xy-1) \subset \mathbb{A}^2_k$ is closed, but maps by f to $\mathbb{A}^1_k \setminus \{0\} \subset \mathbb{A}^1_k$ which is not closed. //

Def A morphism $f: X \to S$ is of *finite presentation* if it is quasi-compact, quasi-separated, and locally of finite presentation."

We proceed to give a category-theoretical characterization of morphisms locally of finite presentation. First, a morphism $f: X \to S$ is *affine* if for every affine open $V \subset S$, $f^{-1}(V) \subset X$ is affine open. In particular, affine morphisms are quasi-cpt. and quasi-separated.

Prop Let I be a small filtered category, and let

$$I^{op} \xrightarrow{T} \text{Schemes}$$

be an I^{op}-diagram of schemes. Suppose that for every morphism $\alpha: i \to j$ in I, the corresponding morphism $\alpha^*: T_j \to T_i$ is affine. Then the limit

$$\lim_{I^{op}} T$$

exists.

Pf Easy to prove from today's recitation problems; see EGA IV 8.2.3. //

Prop If $f: X \to S$ is a morphism of schemes, then the following are equivalent.

(1) f is locally of finite presentation.

(2) For every small cofiltered diagram of S-schemes

$$I^{op} \xrightarrow{T} \text{Schemes}/S$$
$$i \longmapsto (T_i \to S)$$

s.t. the schemes T_i are affine, the canonical map

$$\operatorname{colim}_I \operatorname{Hom}_S(T_i, X)$$
$$\longrightarrow \operatorname{Hom}_S(\lim_{I^{op}} T_i, X)$$

is a bijection.

Pf See EGA IV 8.14.2. //

<u>Def</u> A morphism of schemes $f: X \to S$ is <u>smooth</u> (resp. <u>unramified</u>, resp. <u>étale</u>) if it is formally smooth (resp. formally unramified, resp. formally étale) and locally of finite presentation. //

<u>Ex</u> 1) An open immersion is étale.

2) A closed immersion is unramified if and only if it is locally of finite presentation. //

<u>Lemma</u> Smooth, unramified, and étale morphisms are preserved under composition and base-change. //