We say that an A-algebra B is <u>standard smooth</u> if it admits a (finite) presentation of the form

$$B = A[x_1, \ldots, x_{n+d}] / (f_1, \ldots, f_n)$$

s.t. the image in $M_{n, n+d}(B)$ of

$$(\partial f_i / \partial x_j)$$

has maximal rank n.

<u>Thm</u> If $f : X \to S$ is a morphism of schemes, then the following are equivalent.

(1) f is smooth;

(2) for every $x \in X$, there exists affine open neighborhoods $x \in U \subset X$ and $f(x) \in V \subset S$ s.t. $f(U) \subset V$ and s.t. the $\Gamma(V, \mathcal{O}_S)$-algebra $\Gamma(U, \mathcal{O}_X)$ is standard smooth.

Along the way, we will prove that, for $f : X \to S$ smooth, the $\mathcal{O}_X$-module is locally free of finite rank. This rank, which is locally constant on X is called the <u>relative dimension</u> of f.

We recall that a ring homomorphism $f: A \to B$ is of finite presentation if and only if the induced map of affine schemes $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is locally of finite presentation. Hence, it will suffice to prove the following result.

Prop If $f: A \to B$ is a ring homomorphism of finite presentation, then the following are equivalent.

(1) f is formally smooth;

(2) there exist a covering of $\mathrm{Spec}(B)$ by distinguished open subsets $D(g)$ such that each B_g is a standard smooth A-algebra. /

To prove the proposition, we first characterize formal smoothness of ring homomomorphisms in different ways. First, if $P = A[x_i \mid i \in I]$ is a polynomial algebra over A, then the unique A-algebra homomorphism $f: A \to P$ is formally smooth. Indeed, a lifting in the following diagram is obtained by choosing lifts in $\tilde{C}$ of the images of the x_i in C.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \tilde{C} \arrow[d] \\ P \arrow[r] \arrow[ur, dashed] & C \end{tikzcd}$$

<u>Lemma</u> Let $f : A \to B$ be a ring homomorphism, let $g : P \to B$ be an A-algebra homomorphism from a polynomial algebra onto B, and let $J \subset P$ be the kernel of g. The following are equivalent.

(1) $f : A \to B$ is formally smooth;

(2) the A-algebra homomorphism

$$P/J^2 \longrightarrow B$$

induced by g admits a section.

<u>Pf</u> (1) $\Rightarrow$ (2) follows from

$$\begin{tikzcd} A \arrow[r] \arrow[d] & P/J^2 \arrow[d] \\ B \arrow[r, equal] \arrow[ur, dashed, "\exists"] & B \end{tikzcd}.$$

(2) ⇒ (1) : Consider a diagr.

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d] & A \arrow[r, equal] \arrow[d] & A \arrow[r] \arrow[d] & \tilde{C} \arrow[d, two heads] \\ P \arrow[r, two heads] & P/J^2 \arrow[r, two heads] & B \arrow[r] \arrow[l, bend right, "s"'] & C \end{tikzcd}$$

with $\tilde{C} \twoheadrightarrow C$ a surjection with square-zero kernel I. Since $A \to P$ is formally smooth, there exists $h : P \to \tilde{C}$ lifting $P \to C$. Hence, $h(J) \subset I$, and since $I^2 = 0$, h factors through $\bar{h} : P/J^2 \to \tilde{C}$. Now the composite A-algebra homomorphism

$$B \xrightarrow{s} P/J^2 \xrightarrow{\bar{h}} \tilde{C}$$

is the desired lifting. //

<u>Lemma</u> Let $f : A \to B$ be a ring homomorphism, let $g : P \to B$ be an A-algebra homomorphism onto B, and let J be the kernel. The following are equivalent.

(1) f is formally smooth;

(2) the sequence of B-modules

$$0 \to J/J^2 \to B \otimes_P \Omega_{P/A} \to \Omega_{B/A} \to 0$$

B split-exact

Pf (1) $\Rightarrow$ (2) : For any $n \geq 1$, the canonical projection $P \to P/J^{n+1}$ induces an isomorphism

$$(P/J^n) \otimes_P \Omega_{P/A}$$

$$\xrightarrow{\sim} (P/J^n) \otimes_{(P/J^{n+1})} \Omega_{(P/J^{n+1})/A} .$$

In the case $n=1$, we have

$$B \otimes_P \Omega_{P/A}$$

$$\xrightarrow{\sim} B \otimes_{(P/J^2)} \Omega_{(P/J^2)/A} .$$

By the previous lemma, we have an A-algebra isomorphism

$$B \oplus J/J^2 \xrightarrow[\sim]{s+i} P/J^2$$

with the mult. on the left-hand side given by

$$(b, x) \cdot (b', x') = (bb', bx' + b'x)$$

from which we obtain an isom. of B-modules

$$\Omega_{B/A} \oplus J/J^2$$

$$\xrightarrow{\sim} B \underset{(B \oplus J/J^2)}{\otimes} \Omega_{(B \oplus J/J^2)/A}.$$

So (2) follows.

(2) $\Rightarrow$ (1). Choose B-module section

$$B \otimes_P \Omega_{P/A} \overset{\sigma}{\underset{}{\rightleftarrows}} \Omega_{B/A}$$

and choose a set section

$$\begin{array}{ccc} P & \overset{}{\underset{g}{\rightleftarrows}} & B \\ \ni & & \ni \\ x_b & \longleftarrow & b \end{array}$$

Finally, for every $b \in B$, choose $y_b \in J$ s.t.

$$dy_b = dx_b - \sigma(db).$$

We claim that the map

$$B \xrightarrow{s} P/J^2$$

$$b \longmapsto x_b - y_b + J^2$$

is an A-algebra section of $\bar{g}$. To prove this, we use that, by exactness (on the left) of the sequence in (2), if $h \in J$ and $1 \otimes dh = 0$ in $B \otimes_P \Omega_{P/A}$, then $h \in J^2$. First, let $b_1, b_2 \in B$.

$$d(b_1 + b_2) - db_1 - db_2 = 0 \quad \Rightarrow$$

$$\sigma(d(b_1+b_2)) - \sigma(db_1) - \sigma(db_2) = 0 \quad \Rightarrow$$

$$d(x_{b_1+b_2}) - d(y_{b_1+b_2}) - d(x_{b_1}) \quad \Rightarrow$$
$$+ d(y_{b_1}) - d(x_{b_2}) + d(y_{b_2}) = 0$$

$$d(x_{b_1+b_2} - y_{b_1+b_2} - x_{b_1} + y_{b_1} - x_{b_2} - y_{b_2}) = 0 \quad \Rightarrow$$

$$x_{b_1+b_2} - y_{b_1+b_2} - x_{b_1} + y_{b_1} - x_{b_2} + y_{b_2} \in J^2 \quad \Rightarrow$$

$$s(b_1+b_2) - s(b_1) - s(b_2) = a.$$

So s is additive. The proofs that s is multiplicative and A-linear are similar. //

<u>Cor</u> If $f: A \to B$ is a smooth ring homomorphism, then $\Omega_{B/A}$ is a finitely generated projective B-module.

Pf Since f is of finite type, we can find a surjective A-algebra homomorphism

$$P = A[x_1, \dots, x_m] \xrightarrow{g} B.$$

By the previous lemma, the sequence of B-modules

$$0 \to J/J^2 \to B \otimes_P \Omega_{P/A} \to \Omega_{B/A} \to 0$$

is split-exact. Here J is the kernel of g. But $B \otimes_P \Omega_{P/A}$ is a free B-module with finite basis $\{dx_1, \dots, dx_m\}$, so $\Omega_{B/A}$ and J/J^2 are f.g. projective B-modules. //

Pf of Prop. (2) ⟹ (1): It suffices to show that if B is a standard smooth A-algebra, then $f: A \to B$ is formally smooth. So choose a presentation

$$P = A[x_1, \dots, x_{n+d}] \overset{g}{\twoheadrightarrow} B$$

s.t. the kernel of g is

$$J = (f_1, \dots, f_n)$$

with

$$g\left(\det\left(\partial f_i/\partial x_j\right)_{1\le i,j\le n}\right) \in B^*$$

We claim that the sequence of B-modules

$$0 \to J/J^2 \to B \otimes_P \Omega_{P/A} \to \Omega_{B/A} \to 0$$

is split-exact. Here, the middle term is a free B-module with basis $\{dx_1, \ldots, dx_{n+d}\}$. Moreover, the B-module map

$$B\{f_1, \ldots, f_n\} \xrightarrow{h} J/J^2$$

from the free B-module with the indicated basis that takes f_i to $f_i + J^2$ is surjective, and the composite B-module map

$$B\{f_1, \ldots, f_n\} \xrightarrow{h} J/J^2 \to B \otimes_P \Omega_{P/A}$$

maps f_i to

$$df_i = \sum_{j=1}^{n+d} g\left(\frac{\partial f_i}{\partial x_j}\right) \otimes dx_j .$$

Since the Jacobian at the top of the page is invertible, this map is split injective. Hence h is an isomorphism and the sequence is split exact. By the lemma on page 4-5, f is formally smooth. We also see that the image of $\{dx_{n+1}, \ldots, dx_{n+d}\}$ in $\Omega_{B/A}$ is a basis, so the relative dimension is equal to d.

(1) $\Rightarrow$ (2): Next time. /