We recall Nakayama's lemma: let $A$ be a ring, let $I \subset A$ be an ideal, and let $M$ be a finitely generated $A$-module. If $IM = M$, then there exists $f \in 1 + I$ s.t. $fM = 0$.

<u>Lemma</u> let $A$ be a ring, let $P$ be a f.g. projective $A$-module, and let $S \subset P$ be a set of generators. For every $x \in \operatorname{Spec}(A)$, there exists a finite subset $T \subset S$ and an element $f \in A$ s.t.

(1) $x \in D(f)$; and

(2) the $A_f$-module $P_f$ is f.g. free and $T \subset P_f$ is a basis.

<u>Pf</u> The set $S$ generates the $k(x)$-vector space $P \otimes_A k(x)$; we choose $T \subset S$ to be a basis. It is finite, since $P$, and hence $P \otimes_A k(x)$, admits a finite generating set. Let $M$ be the cokernel of the canonical $A$-module homom.

$$A \cdot T \longrightarrow P \longrightarrow M \longrightarrow 0$$

from the free $A$-module generated by $T$ to $P$. It is a f.g. $A$-module and $M \otimes_A k(x) = 0$; whence $\mathfrak{p}_x M = M$. So by Nakayama's lemma, there exists $g \in 1 + \mathfrak{p}_x$ s.t. $gM = 0$, so

$$A_g \cdot T \twoheadrightarrow P_g .$$

The kernel $N$ is a f.g. $A_g$-module, because $A_g \cdot T$ is f.g. and $P_g$ is finitely presented. (Every f.g. projective module is finitely presented.) Moreover, since $P_g$ is a flat $A_g$-module,

$$N \otimes_{A_g} k(x) = 0 .$$

So by Nakayama's lemma, there exists $h \in 1 + \mathfrak{p}_x A_g$ s.t. $hN = 0$. We can choose $h \in 1 + \mathfrak{p}_x$. So with $f = gh$, we have

$$A_f \cdot T \xrightarrow{\sim} P_f$$

and $f \in 1 + \mathfrak{p}_x$ so $x \in D(f)$. //

<u>Rmk</u> In general, neither finitely generated $A$-modules nor finitely presented $A$-modules form an

abelian category; compare Stacks, Lemma 9.5.4. //

Lemma Suppose that an A-algebra B admits a f. presentation

$$B = A[x_1, \dots, x_m]/(f_1, \dots, f_n)$$

s.t. the B-module

$$(f_1, \dots, f_n)/(f_1, \dots, f_n)^2$$

is free. Then there exists a f. presentation

$$B = A[y_1, \dots, y_k]/(g_1, \dots, g_\ell)$$

s.t. the B-module

$$(g_1, \dots, g_\ell)/(g_1, \dots, g_\ell)^2$$

is free and s.t. the subset

$$\{g_1 + (g_1, \dots, g_\ell)^2, \dots, g_\ell + (g_1, \dots, g_\ell)^2\}$$

is a basis.

Pf Let $g_1, \dots, g_{\ell-1} \in (f_1, \dots, f_n) = I$ be elements s.t.

$$\{g_1 + I^2, \dots, g_{\ell-1} + I^2\} \subset I/I^2$$

is a $B$-module basis, and let

$$M = I/(g_1, \dots, g_{\ell-1}).$$

Since $M \otimes_A A/I = 0$, we have $IM = M$, so by Nakayama's lemma, there exists $h \in 1 + I$ s.t. $h \cdot M = 0$. It follows that

$$h \cdot I \subset (g_1, \dots, g_{\ell-1}).$$

Hence, the canonical projection induces an isomorphism

$$(A[x_1, \dots, x_m]/(g_1, \dots, g_{\ell-1}))[1/h]$$

$$\xrightarrow{\sim} B.$$

Let $k = m+1$, $y_1 = x_1, \dots, y_{k-1} = x_m$, and $g_\ell = h \cdot y_\ell - 1$. Then

$$B = A[y_1, \dots, y_k]/(g_1, \dots, g_\ell)$$

and the image of $\{g_1, \dots, g_\ell\}$ in $(g_1, \dots, g_\ell)/(g_1, \dots, g_\ell)^2$ is a basis as desired. //

We return to the proof of the Prop. on page 2 in lecture 12.

Pf (1) $\Rightarrow$ (2): We choose a f. pres.

$$\begin{array}{ccccc} I & \longrightarrow & P & \xrightarrow{\alpha} & B \\ \| & & \| & & \\ (f_1,\dots,f_n) & & A[x_1,\dots,x_m] & & \end{array}$$

of $f: A \to B$. Since $f$ is formally smooth, the sequence of $B$-mod.

$$0 \to I/I^2 \to B \otimes_P \Omega_{P/A} \to \Omega_{B/A} \to 0$$

is split-exact; hence, $I/I^2$ and $\Omega_{B/A}$ are f.g. projective $B$-modules. We now fix $x \in \operatorname{Spec}(B)$ and use the lemma on page 1 to find $g \in B$ s.t. $x \in D(g)$ and s.t. $(I/I^2)_g$ is a f.g. free $B$-module. We choose $\tilde{g} \in P$ with $\alpha(\tilde{g}) = g$ and consider the presentation

$$\begin{array}{ccccc} I' & \longrightarrow & P' & \longrightarrow & B_g \\ \| & & \| & & \\ (f_1,\dots,f_n,\tilde{g}x_{m+1}-1) & & A[x_1,\dots,x_m,x_{m+1}] & & \end{array}$$

The $B$-module isomorphism

$$(I/I^2)_g \oplus B_g \cdot (\tilde{g} x_{m+1} - 1) \xrightarrow{\sim} I'/I'^2$$

shows that $I'/I'^2$ is a free $B_g$-module. Hence, by the lemma on page 3, there exists a presentation

$$\begin{array}{ccccc} J & \longrightarrow & Q & \xrightarrow{\beta} & B_g \\ \| & & \| & & \\ (g_1, \dots, g_e) & & A[y_1, \dots, y_k] & & \end{array}$$

s.t. the image of $\{g_1, \dots, g_e\}$ in $J/J^2$ is a $B_g$-module basis. We now consider the split-exact sequence of $B_g$-modules

$$0 \to J/J^2 \to B_g \otimes_Q \Omega_{Q/A} \to \Omega_{B_g/A} \to 0.$$

The subset $S = \{dy_1, \dots, dy_e\}$ of the middle term is a basis, and we now choose a subset $T \subset S$ (of cardinality $\ell$) s.t. the composition

$$J/J^2 \longrightarrow B_g \otimes_Q \Omega_{Q/A} \longrightarrow B_g \cdot T$$

of the left-hand map in the sequence above and the canonical projection induces an isomorphism of $k(x)$-vector spaces

$$J/J^2 \otimes_{B_g} k(x) \xrightarrow{\sim} k(x)\cdot T$$

Arguing as in the proof of the lemma on page 1, we may find $h \in B$ such that $x \in D(h)$ and such the map on the previous page induces an isomorphism of $B_{gh}$-modules

$$(J/J^2)_h \xrightarrow{\sim} B_{gh}\cdot T.$$

We choose $\tilde{h} \in P$ with $\alpha(\tilde{h}) = h$ and consider the presentation

$$\begin{array}{ccccc} K & \longrightarrow & R & \xrightarrow{\gamma} & B_{gh}, \\ \parallel & & \parallel & & \\ (g_1,\dots,g_\ell,\tilde{h}y_{k+1}-1) & & A[y_1,\dots,y_k,y_{k+1}] & & \end{array}$$

where $\gamma$ extends $\beta$ and maps $y_{k+1}$ to $1/h$, and consider the split-exact sequence of $B_{gh}$-modules

$$0 \to K/K^2 \to B_{gh} \otimes_R \Omega_{R/A} \to \Omega_{B_{gh}/A} \to 0$$

Let $U$ be the subset $T \cup \{dy_{\ell+1}\}$ of the basis $S \cup \{dy_{\ell+1}\}$ of the middle term. Then the composition

$$K/K^2 \longrightarrow B_{gh} \otimes_R \Omega_{R/A} \longrightarrow B_{gh}\cdot U$$

is an isomorphism of $B_{gh}$-modules. Let $g_{l+1} = \tilde{h} y_{k+1} - 1$. Then the composite map takes the basis element $g_i + K^2$, $1 \leq i \leq l+1$, to

$$\sum_{j \in U} \gamma\left(\frac{\partial g_i}{\partial y_j}\right) \cdot dy_j .$$

We therefore conclude that the $(l+1) \times (l+1)$-minor in the matrix

$$\left(\gamma\left(\partial g_i / \partial y_j\right)\right)_{1 \leq i \leq l+1,\, 1 \leq j \leq k+1}$$

defined by $U$ is invertible, so the whole matrix has maximal rank $l+1$. This shows that $B_{gh}$ is a standard smooth $A$-algebra. In addition, $x \in D(gh) = D(g) \cap D(h)$, so (2) holds. //

The theorem from lecture 12, which we have now finished proving, has the following very useful corollary. It is useful, since it separates proving some property for smooth morphisms into proving said property for (a) polynomial extensions and (b) étale morphisms.

Cor A morphism $f : X \to S$ is smooth if and only if for every $x \in X$, there exists (affine) open neighborhoods $x \in U \subset X$ and $f(x) \in V \subset S$ and an étale morphism $g : U \to \mathbb{A}^d_V$ s.t.

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[d, "f|U"'] & \mathbb{A}^d_V \arrow[d, "pr"] \\ V \arrow[r, equal] & V \end{tikzcd}$$

commutes.

Pf Suppose that $f : A \to B$ is a standard smooth $A$-algebra of relative dimension $d$. Possibly reordering the variables, we have

$$B = A[x_1, \ldots, x_{n+d}] / (f_1, \ldots, f_n)$$

with the image in $M_n(B)$ of

$$(\partial f_i / \partial x_j)_{1 \le i,j \le n}$$

invertible. This shows that $f$ is the composition of the unique $A$-algebra homomorphism

$$A \to A[x_{n+1}, \ldots, x_{n+d}] = B$$

and the (standard) étale $B$-algebra homomorphism

$$B \to B[x_1, \dots, x_n]/(f_1, \dots, f_n).$$

The corollary now follows from the theorem from lecture 12. //

Rmk An étale morphism need not be separated. If $f \colon X \to S$ is a morphism locally of finite presentation, then $f$ is unramified if and only if the diagonal morphism

$$X \xrightarrow{\Delta_{X/S}} X \times_S X$$

is an open immersion. However, the diagonal need not be a closed immersion, even if $f$ is étale. //

Prop Let $f \colon Y \to X$ and $g \colon X \to S$ be morphisms of schemes and assume that $g$ is unramified. If $g \circ f$ is smooth (resp. unramified, resp. étale), then so is $f$.

Pf We first show that if $g$ and $g \circ f$ are locally of finite presentation, then so is $f$. Since $g$ is (in particular) locally of finite type, $\Delta_{X/S} : X \to X \times_S X$ is locally of finite presentation. The cartesian square

$$\begin{tikzcd} Y \arrow[r, "\Gamma_f"] \arrow[d, "f"'] & Y \times_S X \arrow[d, "f \times id"] \\ X \arrow[r, "\Delta_{X/S}"] & X \times_S X \end{tikzcd}$$

shows that also the graph of $f$, $\Gamma_f : Y \to Y \times_S X$, is locally of finite presentation. Now, we may write $f$ as the composition

$$Y \xrightarrow{\Gamma_f} Y \times_S X \xrightarrow{pr_2} X ,$$

and $pr_2$ is locally of finite presentation, since the square

$$\begin{tikzcd} Y \times_S X \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g \circ f"] & S \end{tikzcd}$$

is cartesian and since $g \circ f$ is locally of finite presentation. This shows that $f$ is locally of finite presentation.

Suppose now that $g \circ f$ is formally smooth and that $g$ is formally unramified. In the diagram

$$\begin{tikzcd}
T \arrow[r, "h'"] \arrow[d, hook, "i"] & Y \arrow[d, "f"] \\
\tilde{T} \arrow[r, "h"] \arrow[d, equal] & X \arrow[d, "g"] \\
\tilde{T} \arrow[r] \arrow[uur, dashed, "k"] & S
\end{tikzcd}$$

with $T \hookrightarrow \tilde{T}$ an infinitesimal thickening of affine schemes, a lifting exists, since $g \circ f$ is formally smooth. Hence $f \circ k$ and $h$ are both liftings in the lower square, and since $g$ is formally unramified, they agree. So we have $f \circ k = h$ and $k \circ i = h'$, which proves that $f$ is formally smooth.

Finally, if $g \circ f$ is formally unramified, then $\Omega_{Y/S} = 0$. The exact sequence of $\mathcal{O}_Y$-modules

$$\mathcal{O}_Y \otimes_{f^*\mathcal{O}_X} \Omega_{X/S} \longrightarrow \Omega_{Y/S} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

shows that also $\Omega_{Y/X} = 0$ which, in turn, shows that $f$ is formally unramified. //

Ex We consider a finite ext. of number fields and the corresponding extension of rings of integers:

$$\begin{array}{ccccccc} L & \supset & \mathcal{O}_L & \qquad & \mathrm{Spec}(\mathcal{O}_L) & = & Y \\ | & & | & & \downarrow f & & \downarrow f \\ K & \supset & \mathcal{O}_K & & \mathrm{Spec}(\mathcal{O}_K) & = & X \end{array}$$

Let $x \in \mathrm{Spec}(\mathcal{O}_K)$, let

$$f^{-1}(x) = \{y_1, \dots, y_r\} \subset \mathrm{Spec}(\mathcal{O}_L),$$

and let $\pi_i \in \mathfrak{m}_{y_i} \subset \mathcal{O}_{L,y_i}$ be a a choice of generator. Then we have a cartesian square

$$\begin{tikzcd} \coprod_{i=1}^{r} \mathrm{Spec}(k(y_i)[\pi_i]/(\pi_i^{e_i})) \arrow[r, "\Sigma y_i"] \arrow[d] & \mathrm{Spec}(\mathcal{O}_L) \arrow[d, "f"] \\ \mathrm{Spec}(k(x)) \arrow[r, "x"] & \mathrm{Spec}(\mathcal{O}_K) \end{tikzcd}$$

where $e_i$ is the ramification index of $f$ at $y_i$.

In general, if $f : Y \to X$ is a morphism locally of finite presentation, then the annihilator ideal $\partial_{Y/X} \subset \mathcal{O}_Y$ of $\Omega_{Y/X}$ is called the <u>different</u> of $f$, and the closed subscheme of $Y$ that it defines is called the <u>branch locus</u> of $f$. We note that, by definition, the restriction of $f$ to the open complement of the branch locus is an unramified morphism.

In the case at hand, the branch locus is not all of $Y$, hence it is a finite subscheme of $Y$. The restriction of $f$ to the open complement of the branch locus in $Y$ is flat and unramified, hence étale. (It is a general fact, which we have not proved, that a morphism of schemes is étale if and only if it is both flat and unramified.) //