2013.04.29

Two ways to construct new sets out of old ones:

- as solution sets to systems of equations

  $\rightsquigarrow$ limits

- by gluing sets together

  $\rightsquigarrow$ colimits

Let I be an (index) category. An I-diagram in a category $\mathcal{C}$ is a functor $X : I \to \mathcal{C}$.

Ex 1) If I is a discrete category (i.e. only morphisms in I are the identity morphisms), then an I-diagr. $X : I \to \mathcal{C}$ determines and is determined by the set of objects $\{X_i\}_{i \in ob(I)}$.

2) If I has a single object $*$ and $End_I(*) = Aut_I(*)$ is a group G with composition

$$g \circ h = gh,$$

then an I-diagram $X : I \longrightarrow \mathcal{C}$ determines and is determined by an object $X(*)$ with left G-action:

$$\begin{tikzcd}
* \arrow[d, "h"] \arrow[dd, bend right=60, "gh"'] & & X(*) \arrow[d, "h\cdot"'] \arrow[dd, bend left=60, "gh\cdot"] \\
* \arrow[d, "g"] & \overset{X}{\longmapsto} & X(*) \arrow[d, "g\cdot"'] \\
* & & X(*)
\end{tikzcd}$$

3) If $I = \mathcal{O}(Y)^{op}$ is the opposite of the category of open subsets of a space Y, then an I-diagr.

$$\mathcal{O}(Y)^{op} \xrightarrow{\ F\ } \text{Sets}$$

is, by definition, a presheaf of sets on Y. If $U \subset V \subset W \subset X$, then

$$\begin{tikzcd}
W \arrow[d] \arrow[dd, bend right=60] & & F(W) \arrow[d, "\rho^W_V"'] \arrow[dd, bend left=60, "\rho^W_U"] \\
V \arrow[d] & \longmapsto & F(V) \arrow[d, "\rho^V_U"'] \\
U & & F(U)
\end{tikzcd}$$

A <u>limit</u> of $X : I \longrightarrow \mathcal{C}$ is a set

$$\left\{ \lim_I X \xrightarrow{pr_i} X_i \right\}_{i \in ob(I)}$$

of morphisms in $\mathcal{C}$ with the property that for every morphism $\alpha : i \to j$ in I, the diagram

$$\begin{tikzcd} & X_i \arrow[dd, "X(\alpha)"] \\ \lim_I X \arrow[ur, "pr_i"] \arrow[dr, "pr_j"'] & \\ & X_j \end{tikzcd}$$

commutes and s.t. if

$$\left\{ Y \xrightarrow{f_i} X_i \right\}_{i \in ob(I)}$$

is a another set of morphism with this property, then there exists a unique morphism

$$Y \xrightarrow{f} \lim_I X_i$$

such that

$$f_i = pr_i \circ f$$

for all $i \in ob(I)$.

A <u>colimit</u> of $X : I \to \mathcal{C}$ is a set

$$\{ X_i \xrightarrow{in_i} \operatorname*{colim}_I X \}_{i \in ob(I)}$$

of morphisms in $\mathcal{C}$ with the property that for every morph. $\alpha : i \to j$ in I, the diagram

$$\begin{tikzcd} X_i \arrow[d, "X(\alpha)"'] \arrow[dr, "in_i"] & \\ X_j \arrow[r, "in_j"'] & \operatorname*{colim}_I X \end{tikzcd}$$

commutes and s.t. if

$$\{ X_i \xrightarrow{f_i} Y \}_{i \in ob(I)}$$

is another set of morphisms with this property, then there is a unique morphism

$$\operatorname*{colim}_I X \xrightarrow{f} Y$$

s.t.

$$f_i = f \circ in_i$$

for all $i \in ob(I)$.

Notation: We write

$$Y \xrightarrow{(f_i)} \lim_I X$$

for the unique morphism determined by $\{ f_i : Y \to X_i \}$ and

$$\operatorname{colim}_I X \xrightarrow{\Sigma f_i} Y$$

for the unique morph. determined by $\{ f_i : X_i \to Y \}$.

<u>Rmk</u> Suppose that both

$$\{ \lim_I X \xrightarrow{pr_i} X_i \}_{i \in ob(I)}$$

and

$$\{ \lim_I{}' X \xrightarrow{pr_i'} X_i \}_{i \in ob(I)}$$

are limits of $X : I \to \mathcal{C}$. Then

$$\lim_I X \underset{(pr_i)}{\overset{(pr_i')}{\rightleftarrows}} \lim_I X$$

are each others inverses. So the limit is well-defined, up to unique isomorphism. Similarly for the colimit. //

The limit of a diagr. $X: I \to$ Sets of sets is given by the subset

$$\lim_I X \subset \prod_{i \in ob(I)} X_i$$

consisting of the tuples $(x_i)_{i \in ob(I)}$ s.t. for every morph. $\alpha: i \to j$ in I,

$$X(\alpha)(x_i) = x_j .$$

The map $pr_i : \lim_I X \to X_i$ reads off the i'th component x_i.

Dually, the colimit of $X: I \to$ Sets is given by the quotient

$$\operatorname{colim}_I X_i = \left(\coprod_{i \in ob(I)} X_i\right) / \approx$$

where $\approx$ is the equivalence relation generated by the relation that identifies $x_i \in X$ and $x_j \in X_j$ if there exists a morph. $\alpha: i \to j$ in I s.t. $X(\alpha)(x_i) = x_j$. So, in general,

$$x_i \in X_i \quad \approx \quad x_j \in X_j$$

if there exists morphisms in I

$$\begin{tikzcd}[column sep=small]
& j_1 & & & \cdots & & j_n & \\
i \arrow[ur] & & i_1 \arrow[ul] \arrow[r] & {} & & i_{n-1} \arrow[ul] \arrow[ur] & & j \arrow[ul]
\end{tikzcd}$$

such that the induced maps

$$\begin{tikzcd}[column sep=small]
& X_{j_1} & & & \cdots & & X_{j_n} & \\
X_i \arrow[ur] & & X_{i_1} \arrow[ul] \arrow[r] & {} & & X_{i_{n-1}} \arrow[ul] \arrow[ur] & & X_j \arrow[ul]
\end{tikzcd}$$

take

$$\begin{tikzcd}[column sep=small]
& x_{j_1} & & & \cdots & & x_{j_{n-1}} & \\
x_i \arrow[ur, mapsto] & & x_{i_1} \arrow[ul, mapsto] \arrow[r, mapsto] & {} & & x_{i_{n-1}} \arrow[ul, mapsto] \arrow[ur, mapsto] & & x_j \arrow[ul, mapsto]
\end{tikzcd}$$

In general, the equivalence relation $\approx$ is very difficult to understand. It is much more well-behaved, however, if the category I is <u>filtered</u>, i.e.

(i) $I \neq \emptyset$;

(ii) for every pair of objects $i, j \in ob(I)$, there exists a pair of morphisms

$$\begin{tikzcd}
i \arrow[dr] & \\
& k \\
j \arrow[ur] &
\end{tikzcd}$$

to a common target,

(iii) for every pair of parallel morph.

$$i \overset{\alpha}{\underset{\beta}{\rightrightarrows}} j ,$$

there exists a morph. $\gamma : j \to k$ s.t. $\gamma \circ \alpha = \gamma \circ \beta : i \to k$.

Ex In examples above, (1) and (2) are not filtered except in trivial cases, but (3) is filtered. //

Lemma If $X : I \to \mathrm{Sets}$ is a diagr. of sets with I filtered, then

(i) the colimit is given by

$$\operatorname*{colim}_{I} X = \Big(\coprod_{i \in \mathrm{ob}(I)} X_i \Big) / \sim$$

where $\sim$ is the equivalence rel. that identifies $x_i \in X_i \sim x_j \in X_j$ if there exists

$$\begin{array}{ccc} i & & \\ & \searrow^{\alpha} & \\ & & k \\ & \nearrow_{\beta} & \\ j & & \end{array}$$

s.t. $X(\alpha)(x_i) = X(\beta)(x_j) \in X_k$.

(ii) any two elements $x_i \in X_i$ and $x_j \in X_j$ are equivalent under $\sim$

to two elements of the same X_k;

(iii) two elements $x_i, x_i' \in X_i$ are equivalent under $\sim$ if and only if there exists $\alpha: i \to j$ s.t. $X(\alpha)(x_i) = X(\alpha)(x_i') \in X_j$.

Pf The statement (i) is clear once we show that $\sim$ is an equivalence relation. Only transitivity is an issue, and this follows from

$$\begin{tikzcd}[column sep=small] & & \exists\, n & & \\ & l \arrow[ur] & & m \arrow[ul] & \\ i \arrow[ur] & & j \arrow[ul] \arrow[ur] & & k \arrow[ul] \end{tikzcd}.$$

The statement (ii) is immediate from I being filtered. To prove (iii) suppose $x_i \sim x_i' \in X_i$. Then there exist $\alpha, \beta: i \to j$ s.t. $X(\alpha)(x_i) = X(\beta)(x_i')$. Hence, choosing $\gamma: j \to k$ s.t.

$$\gamma \circ \alpha = \gamma \circ \beta =: \delta : i \to k,$$

we have $X(\delta)(x_i) = X(\delta)(x_i') \in X_k$ as desired. //

Let I and J be two (index) categories, and let $X : I \times J \to \mathcal{C}$ be a diagram indexed by the product category. The universal properties of limit and colimit imply that there is a canonical morphism

$$\operatorname*{colim}_I \lim_J X_{ij} \longrightarrow \lim_J \operatorname*{colim}_I X_{ij} .$$

In general, it is not an isomorphism. For example, if $I = J = \emptyset$, then LHS $= \emptyset$ is an initial object in $\mathcal{C}$ while RHS $= *$ is a terminal object in $\mathcal{C}$. We say that J is finite if the sets $\mathrm{ob}(J)$ and $\mathrm{mor}(J)$ both are finite. The following result is extremely useful.

Prop If $X : I \times J \to \mathrm{Sets}$ is a diagram of sets with I filtered and J finite, then the canonical map

$$\operatorname*{colim}_I \lim_J X_{ij} \longrightarrow \lim_J \operatorname*{colim}_I X_{ij}$$

is a bijection.

Pf Finite limits can be obtained by iterated fiber products. For we can clearly get finite products in this way and the equalizer of $f, g : X \rightrightarrows Y$ is the fiber prod.

$$\begin{tikzcd}
X \times_{X \times Y} X \arrow[r, "pr_1"] \arrow[d, "pr_2"] & X \arrow[d, "(id, f)"] \\
X \arrow[r, "(id, g)"] & X \times Y
\end{tikzcd}$$

So it suffices to show that the canonical map

$$\operatorname*{colim}_I (X_i \times_{S_i} Y_i) \longrightarrow (\operatorname*{colim}_I X_i) \times_{(\operatorname*{colim}_I S_i)} (\operatorname*{colim}_I Y_i)$$

$$\text{class of } (x_i, y_i) \longmapsto (\text{class of } x_i, \text{class of } y_i)$$

is a bijection. Construct inverse map. Let $(\bar{x}, \bar{y})$ be an element of RHS. If $x_i \in X_i$ represents $\bar{x}$ and $y_j \in Y_j$ represents $\bar{y}$, then

$$f(x_i) \in S_i \sim g(y_j) \in S_j .$$

We pick $\alpha : i \to k$ and $\beta : j \to k$; then $x_k = X(\alpha)(x_i) \in X_k$ represents $\bar{x}$ and $y_k = Y(\beta)(y_j) \in Y_k$ represents $\bar{y}$, and

$$f(x_k) \sim g(x_k) \in S_k.$$

Finally, we pick $\gamma : k \to m$ s.t.

$$S(\gamma)(f(x_k)) = S(\gamma)(g(x_k)) \in S_m$$

and set $x_m = X(\gamma)(x_k) \in X_m$ and $y_m = Y(\gamma)(y_k) \in Y_m$. Then x_m repr. $\bar{x}$, y_m repr. $\bar{y}$, and

$$(x_m, y_m) \in X_m \times_{S_m} Y_m.$$

Now the map

$$\text{class of } (x_m, y_m) \longleftarrow\!\shortmid (\bar{x}, \bar{y}).$$

is well-defined and inverse to the canonical map. //

Cor The underlying set of a filtered colimit of rings (resp. modules, ...) is the filtered colimit of the underlying diagram of sets.

Pf Let $A : I \to \text{Rings}$ be a filtered diagram of rings. The colimit of the underlying diagram of sets inherits a ring structure:

$$(\operatorname{colim}_I A_i) \times (\operatorname{colim}_I A_i)$$

$$\xleftarrow{\sim} \operatorname{colim}_I (A_i \times A_i)$$

$$\xrightarrow{+,\cdot} \operatorname{colim}_I A_i .$$

Therefore, this <u>is</u> the colimit in the category of rings. //

<u>Ex</u> This is not true if I is not filtered. For example, if $I = \emptyset$, then the colimit is the initial object, which in Sets is $\emptyset$, but in Rings is $\mathbb{Z}$. //

Let X be a space, let $X^\wedge$ be the category of presheaves of sets on X, and let $X^\sim \subset X^\wedge$ be the full subcategory of sheaves.

<u>Thm</u> The canonical inclusion functor

$$X^\wedge \longleftarrow X^\sim$$

admits a left adjoint

$$X^\wedge \xrightarrow{a} X^\sim$$

and a preserves finite limits.

The functor a is called sheafification.
If $\{U_\alpha \to U\}_{\alpha\in A}$ is a covering, define

$$H^0(\{U_\alpha \to U\}, F) \to \prod_{\alpha\in A} F(U_\alpha) \underset{k}{\overset{j}{\rightrightarrows}} \prod_{\alpha,\alpha'\in A} F(U_\alpha \cap U_{\alpha'})$$

to be an equalizer. We define a map of coverings

$$\{U_\alpha \to U\}_{\alpha\in A} \to \{V_\beta \to U\}_{\beta\in B}$$

to be a map $f : A \to B$ s.t.

$$\begin{tikzcd} U_\alpha \arrow[r, hook] \arrow[d] & V_{f(\alpha)} \arrow[d] \\ U \arrow[r, equal] & U \end{tikzcd}$$

It induces

$$H^0(\{V_\beta \to U\}, F) \xrightarrow{f^*} H^0(\{U_\alpha \to U\}, F).$$

Lemma If $f, g : \{U_\alpha \to U\} \to \{V_\beta \to U\}$ are any two maps, then $f^* = g^*$.

Pf Exercise. //

For $U \subset X$ open, let $I(U)$ be the category with objects coverings $\{U_\alpha \to U\}$ of U and with a single morphism $\{U_\alpha \to U\} \to \{V_\beta \to U\}$ is there exists a map of coverings from $\{U_\alpha \to U\}$ to $\{V_\beta \to U\}$. The lemma shows that

$$I(U)^{op} \xrightarrow{H^0(-,F)} \text{Sets}$$

is a well-defined functor, and we define

$$F^+(U) = \operatorname*{colim}_{I(U)^{op}} H^0(-,F)$$

It is a filtered colimit, since

$$\begin{tikzcd}
\{U_\alpha \to U\}_\alpha & \\
& \{U_\alpha \cap V_\beta \to U\}_{\alpha,\beta} \arrow[ul] \arrow[dl] \\
\{V_\beta \to U\}_\beta &
\end{tikzcd}$$

and since there are no parallel morphisms. Moreover, if $U \to V$, then

$$I(U) \longleftarrow I(V)$$

$$\{U \cap V_\alpha \to U\} \longmapsfrom \{V_\alpha \to V\}$$

which, in turn, induces

$$F^+(U) \longleftarrow F^+(V).$$

So F^+ is a presheaf on X. We say that a presheaf on X is <u>separated</u> if for every covering $\{U_\alpha \to U\}_{\alpha \in A}$,

$$F(U) \xrightarrow{i} \prod_{\alpha \in A} F(U_\alpha)$$

is injective. The proofs of the following lemmas are given in Artin's "Grothendieck Topologies."

<u>Lemma</u> For every presheaf F on X, the presheaf F^+ is separated.

<u>Lemma</u> For every separated presheaf F on X, F^+ is a sheaf.

<u>Proof of Theorem</u>: There is a canonical natural map of presheaves $\eta : F \to F^+$ which is an isomorphism, if F is separated. Hence, if $f : F \to G$ is a map of presheaves, the diagram

$$\begin{array}{ccccc}
F & \xrightarrow{\eta} & F^+ & \xrightarrow{\eta} & F^{++} \\
\downarrow f & & \downarrow f^+ & & \downarrow f^{++} \\
G & \underset{\sim}{\xrightarrow{\eta}} & G^+ & \underset{\sim}{\xrightarrow{\eta}} & G^{++}
\end{array}$$

shows that $a(F) = F^{++}$ is left

adjoint to i. By construction, a preserves finite limits. //

Prop (Kan extension) Let $\mathcal{C}$ be a category which admits all limits and colimits, and let $u : I \to J$ be a functor. Then the functor

$$\mathcal{C}^J \xrightarrow{u^*} \mathcal{C}^I$$

$$F \longmapsto F \circ u$$

has both a left adjoint $u_!$ and a right adjoint u_* given by

$$(u_! F)(j) = \operatorname*{colim}_{(i,\, u(i) \to j)} F(i)$$

$$(u_* F)(j) = \lim_{(i,\, j \to u(i))} F(i) .$$

Pf Exercise. //

Let X be a space and let $O(X)$ be the category of open subsets:

$$\text{ob}\, O(X) = \{ U \subset X \text{ open} \}$$

$\text{mor}\, O(X)$: a single morph. $U \to V$ if $U \subset V$.

A continuous map $f: X \to Y$ induces the functor

$$O(X) \xleftarrow{f^{-1}} O(Y).$$

Hence we have functors

$$\begin{tikzcd} X^\wedge \arrow[r, "(f^{-1})^*"] & Y^\wedge \arrow[l, bend right, "(f^{-1})_!"'] \arrow[l, bend left, "(f^{-1})_*"] \end{tikzcd}$$

given by

$$(f_p G)(V) := (f^{-1})^*(G)(V)$$

$$= G(f^{-1}(V))$$

$$(f^p F)(U) := (f^{-1})_!(F)(U)$$

$$= \operatorname*{colim}_{(V,\, U \subset f^{-1}(V))} F(V)$$

The direct image functor f_p preserves sheaves in that

$$\begin{tikzcd} X^\wedge \arrow[r, "f_p"] & Y^\wedge \\ X^\sim \arrow[u, "i_X"] \arrow[r, "f_*"] & Y^\sim \arrow[u, "i_Y"] \end{tikzcd}$$

commutes for a unique functor f_*. The functor f^P, in general, does not preserve sheaves, so we define the <u>inverse</u> <u>image</u> functor

$$X^\sim \xleftarrow{f^*} Y^\sim$$

to be the composite $a_X f^P i_Y$.

<u>Cor</u> The functors (f^*, f_*) form an adjoint pair, and f^* preserves finite limits.

<u>Pf</u> Prove adjointness:

$$\mathrm{Hom}_{X^\sim}(f^* F, G) = \mathrm{Hom}_{X^\sim}(a_X f^P i_Y F, G)$$

$$\xrightarrow{\sim} \mathrm{Hom}_{X^\wedge}(f^P i_Y F, i_X G)$$

$$\xrightarrow{\sim} \mathrm{Hom}_{Y^\wedge}(i_Y F, f_P i_X G)$$

$$= \mathrm{Hom}_{Y^\wedge}(i_Y F, i_Y f_* G)$$

$$= \mathrm{Hom}_{Y^\sim}(F, f_* G).$$

The colimit defining f^P is filtered, so f^P pres. finite limits. So does a_X and i_Y preserves all limits. Hence, f^* pres. finite limits. //