Recall stalk at $x \in X$ of (pre-)sheaf F on X:

$$F_x = i_x^*(F) = \operatorname*{colim}_{U \ni x} F(U).$$

Ex $X = \operatorname{Spec}(A)$, $F = \mathcal{O}_X$:

$$\begin{aligned} \mathcal{O}_{X,x} &= \operatorname*{colim}_{U \ni x} \Gamma(U, \mathcal{O}_X) \\ &\xleftarrow{\sim} \operatorname*{colim}_{D(f) \ni x} \Gamma(D(f), \mathcal{O}_X) \\ &\xleftarrow{\sim} \operatorname*{colim}_{f \notin \mathfrak{p}_x} A_f = A_{\mathfrak{p}_x} \end{aligned}$$

Note that this is a local ring. //

Lemma If $f : F \to G$ is a map of sheaves on X, then the following are equivalent:

(i) f is an isomorphism;

(ii) for all $x \in X$, the induced map

$$F_x \xrightarrow{f_x} G_x$$

is a bijection.

Pf (i) $\Rightarrow$ (ii) is clear, so assume (ii) and prove (i). Let $U \subset X$ be open; we must show that $f_U : F(U) \to G(U)$ is a bijection.

Injectivity: Let $s, s' \in F(U)$ and suppose $f_U(s) = f_U(s') \in G(U)$. Then for all $x \in X$,

$$\begin{tikzcd} F(U) \arrow[r, "f_U"] \arrow[d, "\mathrm{in}_U"] & G(U) \arrow[d, "\mathrm{in}_U"] \\ F_x \arrow[r, "f_x"] & G_x \end{tikzcd} \qquad \begin{tikzcd} s, s' \arrow[r, mapsto] \arrow[d, mapsto] & f_U(s) = f_U(s') \arrow[d, mapsto] \\ s_x, s'_x \arrow[r, mapsto] & f_x(s_x) = f_x(s'_x) \end{tikzcd}$$

By (ii), $s_x = s'_x$ for all $x \in U$. Since F_x is given by a filtered colimit, there exists $x \in V_x \subset U$ s.t.

$$s|_{V_x} = s'|_{V_x}.$$

But $\{V_x\}_{x \in U}$ is a covering of U, and since F is a sheaf (in fact : separated presheaf suffices),

$$F(U) \hookrightarrow \prod_{x \in U} F(V_x)$$

$$s, s' \longmapsto (s|_{V_x}) = (s'|_{V_x})$$

so $s = s'$ as desired.

Surjectivity: Let $t \in G(U)$ and for $x \in U$, set $t_x = \mathrm{in}_U(t) \in G_x$. Use (ii) to choose $s_x \in F_x$ with $f_x(s_x) = t_x$, and use that F_x is given by a filtered colimit to choose $x \in V_x \subset U$ and $s_{V_x} \in F(V_s)$ with $\mathrm{in}_{V_x}(s_{V_x}) = s_x$. Now,

$$\begin{array}{ccc} F(V_x) & \xrightarrow{f_{V_x}} & G(V_x) \\ \downarrow \mathrm{in}_{V_x} & & \downarrow \mathrm{in}_{V_x} \\ F_x & \xrightarrow{f_x} & G_x \end{array}$$

$$\begin{array}{ccccc} s_{V_x} & \longmapsto & f_{V_x}(s_{V_x}) & & t|_{V_x} \\ \downarrow & & \downarrow & & \downarrow \\ s_x & \longmapsto & t_x & = & t_x \end{array}$$

and since G_x is given by a filt. colimit, we can find $x \in W_x \subset V_x$ s.t.

$$f_{W_x}(s_{W_x}) = t|_{W_x}$$

with $s_{W_x} = s_{V_x}|_{W_x}$. The family $\{W_x\}_{x \in U}$ is a covering of U, and for all $x, x' \in U$,

$$s_{W_x}|_{W_x \cap W_{x'}} = s_{W_{x'}}|_{W_x \cap W_{x'}}.$$

Indeed, the map

$$F(W_x \cap W_{x'}) \xrightarrow{f_{W_x \cap W_{x'}}} G(W_x \cap W_{x'})$$

map both to $t|_{W_x \cap W_{x'}}$ and is injective by what was proved earlier. Since F is a sheaf, there thus exists $s \in F(U)$ with $s|_{W_x} = s_{W_x}$, for all $x \in U$. Finally

$$\begin{tikzcd} F(U) \arrow[r, hook] \arrow[d, "f_U"] & \prod_{x \in U} F(W_x) \arrow[d, "\prod f_{W_x}"] \\ G(U) \arrow[r, hook] & \prod_{x \in U} G(W_x) \end{tikzcd} \qquad \begin{tikzcd} s \arrow[r, mapsto] \arrow[d, mapsto] & (s_{W_x}) \arrow[d, mapsto] \\ t \arrow[r, mapsto] & (t|_{W_x}) \end{tikzcd}$$

shows that $f_U(s) = t$. //

Def A __ringed space__ is a pair $(X, \mathcal{O}_X)$ of a space X and a sheaf of rings $\mathcal{O}_X$ on X. A __morphism__ of ringed spaces is a pair

$$(X, \mathcal{O}_X) \xrightarrow{(f, f^\#)} (Y, \mathcal{O}_Y)$$

of a continuous map $f : X \to Y$ and a map $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ of sheaves of rings on Y.

A __locally ringed space__ is a ringed space $(X, \mathcal{O}_X)$ s.t. for every $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring. A morphism of locally ringed spaces is a morphism of ringed spaces

$$(X, \mathcal{O}_X) \xrightarrow{(f, f^\#)} (Y, \mathcal{O}_Y)$$

s.t. for every $x \in X$, the map

$$\mathcal{O}_{Y,y} \xrightarrow{f_x^\#} \mathcal{O}_{X,x} \qquad (y = f(x))$$

is a local ring homomorphism.

That $f_x^\#$ is local means that

$$f_x^\#(m_y) \subset m_x$$

which implies that $f_x^\#$ induces

$$k(y) \xhookrightarrow{\overline{f_x^\#}} k(x).$$

Hence, if $y \in V \subset Y$, the diagram

$$\begin{tikzcd}
\Gamma(V, \mathcal{O}_Y) \arrow[r, "f_V^\#"] \arrow[d, "in_V"] & \Gamma(f^{-1}(V), \mathcal{O}_X) \arrow[d, "in_{f^{-1}(V)}"] \\
\mathcal{O}_{Y,y} \arrow[r, "f_x^\#"] \arrow[d] & \mathcal{O}_{X,x} \arrow[d] \\
k(y) \arrow[r, hook, "\overline{f_x^\#}"] & k(x)
\end{tikzcd}$$

shows that

$$s(y) = 0 \in k(y) \iff$$

$$f_V^{\#}(s)(x) = 0 \in k(x).$$

Ex $\mathbb{Z}_{(p)} \longrightarrow \mathbb{Q}$ is not local, hence, does not induce $\mathbb{F}_p \longrightarrow \mathbb{Q}$. //

Recall that if A is a ring, then

$$(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$$

is a locally ringed space.

Def A locally ringed space $(X, \mathcal{O}_X)$ is a <u>scheme</u> if X admits a covering

$$\{ U_\alpha \xrightarrow{j_\alpha} X \}_{\alpha \in \mathcal{A}}$$

s.t. for all $\alpha \in \mathcal{A}$, $(U_\alpha, j_\alpha^* \mathcal{O}_X)$ is isomorphic as a locally ringed space to $(\mathrm{Spec}(A_\alpha), \mathcal{O}_{\mathrm{Spec}(A_\alpha)})$ for some ring A_α.

The category of schemes is the full subcategory of the category of locally ringed spaces whose objects are the schemes. //

Define functor

$$\begin{array}{ccc} \text{Rings}^{op} & \longrightarrow & \text{Schemes} \\ A' & & (\mathrm{Spec}(A'), \mathcal{O}_{\mathrm{Spec}(A')}) \\ \Big\downarrow \varphi & \longmapsto & \Big\uparrow (f, f^{\#}) \\ A & & (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}) \end{array}$$

Set map

$$\begin{array}{ccc} \mathrm{Spec}(A) & \xrightarrow{f} & \mathrm{Spec}(A') \\ \Psi & & \Psi \\ \mathfrak{p}_x & \longmapsto & \varphi^{-1}(\mathfrak{p}_x) = \mathfrak{p}_{x'} \end{array}$$

Continuous, since if $a = \varphi(a')$, then

$$f^{-1}(D(a'))$$

$$= \{x \in \mathrm{Spec}(A) \mid f(x) \in D(a')\}$$

$$= \{x \in \mathrm{Spec}(A) \mid \varphi^{-1}(\mathfrak{p}_x) \not\ni a'\}$$

$$= \{x \in \mathrm{Spec}(A) \mid \mathfrak{p}_x \not\ni a\}$$

$$= D(a).$$

Map of sheaves:

$$\mathcal{O}_{\mathrm{Spec}(A')} \xrightarrow{f^\#} f_*(\mathcal{O}_{\mathrm{Spec}(A)})$$

induced (by right Kan ext.) from

$$\begin{tikzcd}
\Gamma(D(a'), \mathcal{O}_{\mathrm{Spec}(A')}) \arrow[d, "f^\#_{D(a')}"] & A'_{a'} \arrow[l, "\sim"'] \arrow[ddd, "\varphi"] \\
\Gamma(D(a'), f_*\mathcal{O}_{\mathrm{Spec}(A)}) \arrow[d, equal] & \\
\Gamma(f^{-1}(D(a')), \mathcal{O}_{\mathrm{Spec}(A)}) \arrow[d, equal] & \\
\Gamma(D(a), \mathcal{O}_{\mathrm{Spec}(A)}) & A_a \arrow[l, "\sim"']
\end{tikzcd}$$

If $D(a') \supset D(b')$, then $D(a) \supset D(b)$ and the diagram

$$\begin{tikzcd}
\Gamma(D(a'), \mathcal{O}_{\mathrm{Spec}(A')}) \arrow[r, "\rho_{b',a'}"] \arrow[d, "f^\#_{D(a')}"] & \Gamma(D(b'), \mathcal{O}_{\mathrm{Spec}(A')}) \arrow[d, "f^\#_{D(b')}"] \\
\Gamma(D(a), \mathcal{O}_{\mathrm{Spec}(A)}) \arrow[r, "\rho_{b,a}"] & \Gamma(D(b), \mathcal{O}_{\mathrm{Spec}(A)})
\end{tikzcd}$$

commutes, so $f^\#$ map of sheaves. Moreover, the diagram

$$\begin{tikzcd}
\Gamma(D(a'), \mathcal{O}_{\mathrm{Spec}(A')}) \arrow[r, "\rho_{a'}^{x'}"] \arrow[d, "f^{\#}_{D(a')}"] & A_{\mathfrak{p}_{x'}} \arrow[d, "\varphi"] \\
\Gamma(D(a), \mathcal{O}_{\mathrm{Spec}(A)}) \arrow[r, "\rho_a^x"] & A_{\mathfrak{p}_x}
\end{tikzcd}$$

so $f^{\#}_x$ is a local ring homom. This shows that $(f, f^{\#})$ is a map of locally ringed spaces.

Also have functor

$$\begin{tikzcd}
\text{Schemes} \arrow[r] & \text{Rings}^{op} \\
(X, \mathcal{O}_X) \arrow[d, "(f,f^{\#})"] \arrow[r, mapsto, phantom] & \Gamma(X, \mathcal{O}_X) \\
(Y, \mathcal{O}_Y) & \Gamma(Y, \mathcal{O}_Y) \arrow[u, "f^{\#}_Y"']
\end{tikzcd}$$

Prop The map

$$\mathrm{Hom}_{Sch}((X, \mathcal{O}_X), (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})) \xrightarrow{\alpha} \mathrm{Hom}_{\mathrm{Rings}}(A, \Gamma(X, \mathcal{O}_X))$$

that to $(f, f^{\#})$ associates $f^{\#}_{\mathrm{Spec}(A)}$ is a natural bijection.

Pf Write $(Y, \mathcal{O}_Y) = (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$.

Injectivity: Show $f_Y^\#$ determines $(f, f^\#)$:

$$\begin{tikzcd}
A \arrow[r, "f_Y^\#"] \arrow[d] & \Gamma(X, \mathcal{O}_X) \arrow[d] & a \arrow[r, mapsto] \arrow[dd, mapsto] & f_Y^\#(a) \arrow[dd, mapsto] \\
\mathcal{O}_{Y, f(x)} \arrow[r, "f_x^\#"] \arrow[d] & \mathcal{O}_{X,x} \arrow[d] & & \\
k(f(x)) \arrow[r, hook, "\bar{f}_x^\#"] & k(x) & a(f(x)) \arrow[r, mapsto] & f_Y^\#(a)(x)
\end{tikzcd}$$

shows that

$$\begin{aligned} \mathfrak{p}_{f(x)} &= \{ a \in A \mid a(f(x)) = 0 \} \\ &= \{ a \in A \mid f_Y^\#(a)(x) = 0 \} \end{aligned}$$

so $f_Y^\#$ determines $f : X \to Y$. Next,

$$\begin{tikzcd}
A = \Gamma(Y, \mathcal{O}_Y) \arrow[r, "f_Y^\#"] \arrow[d] & \Gamma(X, \mathcal{O}_X) \arrow[d] \\
A_a = \Gamma(D(a), \mathcal{O}_Y) \arrow[r, "f_{D(a)}^\#"] & \Gamma(f^{-1}(D(a)), \mathcal{O}_X)
\end{tikzcd}$$

localization

so $f_Y^\#$ determines $f_{D(a)}^\#$ for all $a \in A$; hence $f_V^\#$ for all $V \subset Y$ open. This shows $f_Y^\#$ determines $f^\#$.

(We have tacitly assumed that for every ring B, the canonical isom.

$$B \xrightarrow{\sim} \Gamma(\mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}(B)})$$

is the identity map. We will continue to do so for the remainder of this proof.)

Surjectivity: Suppose first

$$(X, \mathcal{O}_X) = (\mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}(B)}).$$

A ring homom. $\varphi : A \to B$ induces a map of schemes

$$(X, \mathcal{O}_X) \xrightarrow{(f, f^\#)} (Y, \mathcal{O}_Y)$$

and, by definition, $f^\#_Y = \varphi$. So this case is OK. Next, if there exists an isomorphism

$$(X, \mathcal{O}_X) \xrightarrow{(g, g^\#)} (\mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}(B)}),$$

then the map $\alpha = \alpha_{(X,A)}$ again is surjective by naturality: the following diagram commutes, the vertical maps are isom., and the lower horizontal map surjective.

$$
\begin{array}{ccc}
\mathrm{Hom}((X,\mathcal{O}_X),(Y,\mathcal{O}_Y)) & \xrightarrow{\alpha} & \mathrm{Hom}(A,\Gamma(X,\mathcal{O}_X)) \\
\sim\uparrow (g,g^\#)^* & & \sim\uparrow g^\#_{\mathrm{Spec}(B)} \\
\mathrm{Hom}((\mathrm{Spec}\,B,\mathcal{O}_{\mathrm{Spec}(B)}),(Y,\mathcal{O}_Y)) & \xrightarrow{\alpha} & \mathrm{Hom}(A,B)
\end{array}
$$

Finally, in the general case, we choose a covering

$$\{ U_\lambda \xhookrightarrow{j_\lambda} X \}$$

s.t. each $(U_\lambda, j_\lambda^* \mathcal{O}_X)$ is isomorphic $(\mathrm{Spec}(B_\lambda), \mathcal{O}_{\mathrm{Spec}(B_\lambda)})$ for some ring B_λ. We also recall from the exercises that if $j: U \hookrightarrow X$ is an open inclusion, then

$$\Gamma(U, j^* \mathcal{O}_X) = \Gamma(U, \mathcal{O}_X).$$

Let $j_{\lambda\mu} : U_\lambda \cap U_\mu \hookrightarrow X$ be the common restriction of j_λ and j_μ. In general, the locally ringed space $(U_\lambda \cap U_\mu, j_{\lambda\mu}^* \mathcal{O}_X)$ is not isom. to $(\mathrm{Spec}(C), \mathcal{O}_{\mathrm{Spec}(C)})$ for some ring C. It is a scheme, however, as we will show in the next lecture. Granting this, we conclude that the diagram

$$\coprod_{\lambda,\mu} (U_\lambda \cap U_\mu, j_{\lambda\mu}^* \mathcal{O}_X) \rightrightarrows \coprod_\nu (U_\nu, j_\nu^* \mathcal{O}_X)$$
$$\longrightarrow (X, \mathcal{O}_X)$$

is a coequalizer in the category of schemes. Now, given a ring homomorphism $\varphi : A \to \Gamma(X, \mathcal{O}_X)$, we define $\varphi_\lambda : A \to \Gamma(U_\lambda, j_\lambda^* \mathcal{O}_X)$ to be the composite map

$$A \xrightarrow{\varphi} \Gamma(X, \mathcal{O}_X)$$
$$\xrightarrow{\rho^X_{U_\lambda}} \Gamma(U_\lambda, \mathcal{O}_X) = \Gamma(U_\lambda, j_\lambda^* \mathcal{O}_X)$$

and let

$$(U_\lambda, j_\lambda^* \mathcal{O}_X) \xrightarrow{(f_\lambda, f_\lambda^\#)} (Y, \mathcal{O}_Y)$$

be the unique map of schemes s.t. $f_Y^\# = \varphi_\lambda$, the existence of which was proved above. Since the diagram of ring homom.

$$\begin{tikzcd}
 & & \Gamma(U_\lambda, \mathcal{O}_X) \arrow[dr, "\rho^{U_\lambda}_{U_\lambda \cap U_\mu}"] & \\
A \arrow[urr, bend left, "\varphi_\lambda"] \arrow[r, "\varphi"] \arrow[drr, bend right, "\varphi_\mu"'] & \Gamma(X, \mathcal{O}_X) \arrow[ur, "\rho^X_{U_\lambda}"] \arrow[dr, "\rho^X_{U_\mu}"'] & & \Gamma(U_\lambda \cap U_\mu, \mathcal{O}_X) \\
 & & \Gamma(U_\mu, \mathcal{O}_X) \arrow[ur, "\rho^{U_\mu}_{U_\lambda \cap U_\mu}"'] &
\end{tikzcd}$$

commutes, the injectivity part of the statement shows that so does the diagram of schemes

$$\begin{tikzcd}
& (U_\lambda, j_\lambda^* \mathcal{O}_X) \arrow[dl, "(f_\lambda, f_\lambda^\#)"'] & \\
(Y, \mathcal{O}_Y) & & (U_\lambda \cap U_\mu, j_{\lambda\mu}^* \mathcal{O}_X) \arrow[ul] \arrow[dl] \\
& (U_\mu, j_\mu^* \mathcal{O}_X) \arrow[ul, "(f_\mu, f_\mu^\#)"] &
\end{tikzcd}$$

Hence, the morphisms $(f_\lambda, f_\lambda^\#)$ glue to give a morphism of schemes

$$(X, \mathcal{O}_X) \xrightarrow{(f, f^\#)} (Y, \mathcal{O}_Y).$$

Moreover, since the composite map

$$A \xrightarrow{f_Y^\#} \Gamma(X, \mathcal{O}_X) \xhookrightarrow{i} \prod_\lambda \Gamma(U_\lambda, \mathcal{O}_X)$$

is equal to (φ_λ), we conclude that $f_Y^\# = \varphi$ as desired. //