 2013.05.06

<u>lemma</u> The canonical map factors through an isomorphism of schemes

$$\begin{tikzcd}
(\mathrm{Spec}(A_f), \mathcal{O}_{\mathrm{Spec}(A_f)}) \arrow[rr] \arrow[dr, "{(h, h^\#)}"'] & & (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}) \\
& (D(f), j^* \mathcal{O}_{\mathrm{Spec}(A)}) \arrow[ur, "{(j, id)}"'] &
\end{tikzcd}$$

<u>Pf</u> Let $\varphi : A \to A_f$ be the localization. The induced map

$$\begin{array}{ccc}
\mathrm{Spec}(A_f) & \longrightarrow & \mathrm{Spec}(A) \\
\psi & & \psi \\
\mathfrak{p} & \longmapsto & \varphi^{-1}(\mathfrak{p})
\end{array}$$

is a bijection onto $D(f)$. In fact, it is a homeomorphism on $D(f)$:

$$\begin{array}{ccc}
\mathrm{Spec}(A_f) & \xrightarrow{h} & D(f) \\
\cup & & \cup \\
D(g) & \xrightarrow{\sim} & D(fg)
\end{array}$$

Isomorphism of ringed spaces:

$$\begin{tikzcd}
\Gamma(D(g), \mathcal{O}_{\mathrm{Spec}(A_f)}) & \Gamma(D(fg), j^* \mathcal{O}_{\mathrm{Spec}(A)}) \arrow[l, "\sim"'] \\
(A_f)_g \arrow[u] & A_{fg} \arrow[l, "\sim"'] \arrow[u, "\sim"']
\end{tikzcd}$$

where the right-hand vertical map is an isom., since $j : D(f) \hookrightarrow X$ is the inclusion of an open subspace. //

<u>Cor</u> Let $(X, \mathcal{O}_X)$ be a scheme, and let $j : U \hookrightarrow X$ be the inclusion of an open subspace. Then the ringed space $(U, j^*\mathcal{O}_X)$ is a scheme. //

We say that a map of schemes

$$(U, \mathcal{O}_U) \xrightarrow{(j, j^\#)} (X, \mathcal{O}_X)$$

is an <u>open immersion</u> if $j : U \to X$ is a homeo. onto an open subset of X and if $j^\# : \mathcal{O}_U \to j^*\mathcal{O}_X$ is an isomorphism.

We proceed to show that colimits corresponding to gluing along open immersions exist in the category of schemes.

<u>Prop</u> The category of ringed spaces has all small colimits.

<u>Pf</u> Given an I-diagram of ringed space $(X, \mathcal{O}_X)$, its colimit is the

is given by the following set of morphisms of ringed spaces.

$$\left\{ \begin{array}{c} (X_i, \mathcal{O}_{X_i}) \\ \downarrow (\mathrm{in}_i, \mathrm{pr}_i) \\ (\operatorname{colim}_I X_i, \lim_I \mathrm{in}_{i*} \mathcal{O}_{X_i}) \end{array} \right\}_{i \in \mathrm{ob}(I)}$$ //

Rmk The forgetful functor from locally ringed spaces to ringed spaces creates all small colimits. This means that if $(X, \mathcal{O}_X)$ is a diagram of locally ringed spaces, then its colimit in the category of ringed spaces IS a locally ringed space and is the colimit in the category of locally ringed spaces. For a proof, see Demazure Gabriel, Groupes algébriques, Prop. I.1.6. //

In the following, we will often abbreviate $(X, \mathcal{O}_X)$ and simply write X; similarly, instead of

$$(X, \mathcal{O}_X) \xrightarrow{(f, f^\#)} (Y, \mathcal{O}_Y),$$

we write $f: X \to Y$. If $(X, \mathcal{O}_X)$ is a scheme and $j: U \hookrightarrow X$ an open subset, we will write U to to indicate the scheme $(U, j^* \mathcal{O}_X)$.

Let I be a (small) set, and let $P_0(I)$ be the following category.

$$\operatorname{ob} P_0(I) = \{\emptyset \neq S \subset I \mid S \text{ finite}\}$$

$\operatorname{mor} P_0(I)$: a single morphism $S \to T$ if $S \subset T$

A diagram of the form

$$P_0(I)^{op} \xrightarrow{X} \mathcal{C}$$

is called a <u>punctured</u> I-<u>cubical diagram</u> in $\mathcal{C}$.

<u>Ex</u> $I = \{1,2\}$:

$$\begin{tikzcd} X_{\{1,2\}} \arrow[r, "f"] \arrow[d, "g"] & X_{\{1\}} \\ X_{\{2\}} & \end{tikzcd}$$

A colimit of this diagram is called a pushout or a cobase change of f along g (or of g along f).

$I = \{1,2,3\}$:

$$\begin{tikzcd}
X_{\{1,2,3\}} \arrow[rr] \arrow[dr] \arrow[dd] & & X_{\{1,2\}} \arrow[dr] \arrow[dd] & \\
& X_{\{1,3\}} \arrow[rr] \arrow[dd] & & X_{\{1\}} \\
X_{\{2,3\}} \arrow[rr] \arrow[dr] & & X_{\{2\}} & \\
& X_{\{3\}} & &
\end{tikzcd}$$

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If $\emptyset \neq S \subset T \subset I$ are finite, then the restriction of the punctured I-cubical diagram to the full subcategory of $P_0(I)$ with object set

$$\{ S \subset U \subset T \}$$

is called the (S,T)-*face* of X and denoted $\partial_S^T X$. For the diagr. at the top of the page,

$$\partial_{\{2\}}^{\{1,2,3\}} X = \begin{tikzcd}
X_{\{1,2,3\}} \arrow[r] \arrow[d] & X_{\{1,2\}} \arrow[d] \\
X_{\{2,3\}} \arrow[r] & X_{\{2\}}
\end{tikzcd}$$

A punctured I-cubical diagram

X is <u>strongly</u> <u>cocartesian</u> if every face of X is cocartesian, or equivalently, if every face $\partial_S^T X$ with $\#(T \setminus S) = 2$ is a cocartesian square.

<u>Ex</u> Let X be a scheme and let $\{U_i\}_{i \in I}$ be a set of open subschemes of X. Then the punctured I-cubical diagram of schemes defined by

$$P_0(I)^{op} \longrightarrow \text{Schemes}$$

$$S \longmapsto U_S := \bigcap_{i \in S} U_i$$

is strongly cocartesian. If the U_i, $i \in I$, cover X, then

$$\{ U_S \hookrightarrow X \}$$

is a colimit of this diagram. //

<u>Prop</u> (Gluing) Every (small) strongly cocartesian punctured cubical diagram of schemes in which every morphism is an open immersion has a colimit.

Pf Let $(X, \mathcal{O}_X)$ be a punctured I-cubical diagram of the form stated, and let

$$\{ (X_S, \mathcal{O}_{X_S}) \xrightarrow{(in_S, in_S^\#)} (Y, \mathcal{O}_Y) \}$$

be a colimit in the category of ringed spaces. We must show that $(Y, \mathcal{O}_Y)$ is a scheme, that the maps $(in_S, in_S^\#)$ are morph. of schemes, and that the set of these morphisms is a colimit in the category of schemes.

First, the space Y is the quotient

$$Y = \Big(\coprod_{i \in I} X_{\{i\}} \Big) / \sim$$

where $\sim$ is the equivalence rel.

$$x_i \in X_{\{i\}} \sim x_j \in X_{\{j\}} \overset{}{\underset{\text{def}}{\Longleftrightarrow}}$$

$$\begin{tikzcd} X_{\{i\}} & & X_{\{j\}} \\ & X_{\{i,j\}} \arrow[ul] \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} x_i & & x_j \\ & \exists\, x_{ij} \arrow[ul, mapsto] \arrow[ur, mapsto] & \end{tikzcd}$$

To see that $\sim$ is transitive, we consider the diagram

$$\begin{tikzcd}[column sep=small]
X_{\{i\}} & & X_{\{j\}} & & X_{\{k\}} \\
& X_{\{i,j\}} \arrow[ul] \arrow[ur, hook] & & X_{\{j,k\}} \arrow[ul] \arrow[ur, hook] & \\
& & X_{\{i,j,k\}} \arrow[ul] \arrow[ur, hook] & &
\end{tikzcd}$$

The middle square is a cocartesian diagram of spaces in which the maps are open inclusions; but such a square is also cartesian. Thus

$$\begin{tikzcd}[column sep=small]
x_{i} & & x_{j} & & x_{k} \\
& x_{ij} \arrow[ul, mapsto] \arrow[ur, mapsto] & & x_{jk} \arrow[ul, mapsto] \arrow[ur, mapsto] & \\
& & \exists\, x_{ijk} \arrow[ul, mapsto] \arrow[ur, mapsto] & &
\end{tikzcd}$$

and the element

$$X_{\{i,j,k\}} \hookrightarrow X_{\{i,k\}}$$

$$x_{ijk} \longmapsto x_{ik}$$

shows that $x_i \sim x_k$.

It follows that $in_S : X_S \longrightarrow Y$ is an open inclusion. We leave it as an exercise to show that

$$in_S^* \mathcal{O}_Y \xrightarrow{in_S^\#} \mathcal{O}_{X_S}$$

is an isomorphism. Since the open subsets $in_S(X_S) \subset Y$ cover Y, this shows that $(Y, \mathcal{O}_Y)$ is a scheme and that

$$(X_S, \mathcal{O}_{X_S}) \xrightarrow{(in_S, in_S^\#)} (Y, \mathcal{O}_Y)$$

is a morphism of schemes; in fact, that it is an open immersion. Finally, let

$$\{ (X_S, \mathcal{O}_{X_S}) \xrightarrow{(f_S, f_S^\#)} (Z, \mathcal{O}_Z) \}$$

be a set of morphisms of schemes, compatible with the morphisms in the cubical diagram. We must show that the unique morphism of ringed spaces

$$(Y, \mathcal{O}_Y) \xrightarrow{(f, f^\#)} (Z, \mathcal{O}_Z)$$

s.t.

$$(f_S, f_S^\#) = (f, f^\#) \circ (in_S, in_S^\#)$$

for all S, is a morphism of locally ringed spaces. That is, if $y \in Y$ and $z = f(y) \in Z$, then we must show that

$$\mathcal{O}_{Z,z} \xrightarrow{f^{\#}_y} \mathcal{O}_{Y,y}$$

is local. But if $y \in in_S(X_S)$, $y = in_S(x)$, then the diagram

$$\begin{tikzcd} \mathcal{O}_{Z,z} \arrow[rr, "f^{\#}_y"] \arrow[dr, "f^{\#}_{S,x}"'] & & \mathcal{O}_{Y,y} \arrow[dl, "in^{\#}_{S,x}"] \\ & \mathcal{O}_{X_S,x} & \end{tikzcd}$$

commutes, and $in^{\#}_{S,x}$ is an isom., since $(in_S, in^{\#}_S)$ is an open immersion. Hence, as $f^{\#}_{S,x}$ is local, so is $f^{\#}_y$. //

Ex (1) Let A be a ring, and let $I = A$. There is a strongly cocart. punctured I-cubical diagram with

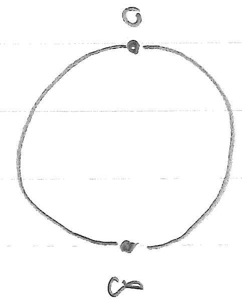
$$P_0(A)^{op} \longrightarrow \text{Schemes}$$

$$S \longmapsto (\mathrm{Spec}(S^{-1}A), \mathcal{O}_{\mathrm{Spec}(S^{-1}A)})$$

and with the morphisms induced by the canonical ring homomorphisms $S^{-1}A \longrightarrow T^{-1}A$. The colimit of course is $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$.

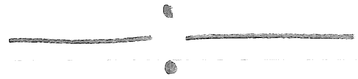
(2) The projective line is obtained as the following colimit

$$\begin{tikzcd} \mathrm{Spec}(k[t^{\pm 1}]) \arrow[r, "t \leftarrow\!\shortmid x"] \arrow[d, "y \mapsto t^{-1}"] & \mathrm{Spec}(k[x]) \arrow[d] \\ \mathrm{Spec}(k[y]) \arrow[r] & \mathbb{P}^1_k \end{tikzcd}$$



(3) The affine line with a double point is the following colimit

$$\begin{tikzcd} \mathrm{Spec}(k[t^{\pm 1}]) \arrow[r, "t \leftarrow\!\shortmid x"] \arrow[d, "y \mapsto t"] & \mathrm{Spec}(k[x]) \arrow[d] \\ \mathrm{Spec}(k[y]) \arrow[r] & X \end{tikzcd}$$



(4) Let $\mathcal{O}$ be a noetherian local ring, let $X = \mathrm{Spec}\,\mathcal{O}$, and let

$$U = X \setminus \{x\}.$$

We claim that $U \subset X$ is open, or

equivalently, that $\{x\} \subset X$ is closed. In general, $\overline{\{x\}} = V(\mathfrak{p}_x)$, since

$$x \in V(I) \iff \mathfrak{p}_x \supset I \implies V(\mathfrak{p}_x) \subset V(I),$$

and

$$\{x\} = \overline{\{x\}} \iff \mathfrak{p}_x = \{\mathfrak{p} \subset A \mid \mathfrak{p} \supset \mathfrak{p}_x\} \iff$$

$$\mathfrak{p}_x \subset A \text{ maximal.}$$

So $U \subset X$ is open. If

$$\mathfrak{m}_x = \sqrt{(f_1, \dots, f_n)},$$

then

$$\begin{aligned} \{x\} &= V(\mathfrak{m}_x) \\ &= \{y \in X \mid f_1(y) = \dots = f_n(y) = 0\} \\ &= \bigcap_{i=1}^{n} \{y \in X \mid f_i(y) = 0\} \end{aligned}$$

so

$$\begin{aligned} U &= \bigcup_{i=1}^{n} D(f_i) \\ &= \bigcup_{i=1}^{n} \operatorname{Spec}(\mathcal{O}_{f_i}). \end{aligned}$$

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