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In standard ZFC set theory, the only undefined term is "set" and sets satisfy a list of axioms (and axiom schemes). In other words, every mathematical object is a set. For example, an <u>ordinal</u> is defined to be a hereditarily transitive set. Here, transitive means that $y \in x$ and $z \in y$ implies $z \in x$, and that this property is hereditary means that it also holds for all elements of the set, all elements of elements, and so on. (Incidentally, only finite chains of sets

$$x_n \in x_{n-1} \in \cdots \in x_2 \in x_1 \in x_0$$

exist, by the axiom of foundation. In particular, $x \in x$ is not allowed.) If α is an ordinal, then the <u>successor ordinal</u> is defined by

$$\alpha + 1 = \alpha \cup \{\alpha\}.$$

An ordinal α is a <u>limit ordinal</u> if

$$\alpha = \bigcup_{\beta \in \alpha} \beta.$$

Note, however, that we cannot simply

define this α to be the union of all smaller ordinals, because we only know (by the axiom of union and the axiom scheme of replacement) that a family of sets indexed by a set has a union. (A cardinal is defined to be an ordinal that is not bijective to any of its elements.) So how many ordinals are there? This depends on our model of ZFC. While ZFC guarantees the existence of some ordinals (e.g. the first infinite ordinal ω but many more), we may wish to have more. Large cardinal axioms are axioms that postulate the existence of certain ordinals (and cardinals), whose existence does not follow from ZFC. The axiom of universe, which we now explain, is of this kind.

Def A set U is a universe if it has the following five properties.

(U1) If $x \in U$ any $y \in x$, then $y \in U$.

(U2) If $x, y \in U$, then $\{x, y\} \in U$.

(U3) If $x \in U$, then $\mathcal{P}(x) \in U$.

(U4) If $I \in U$ and $x : I \longrightarrow U$ is a map, then $\bigcup_{i \in I} x(i) \in U$.

(U5) $\omega \in U$. "

We will assume:

~~Axiom~~ of Universe: For every set x, there exists a universe U s.t. $x \in U$.

If U is a universe, then the sets $x \in U$ form a model of ZFC. (The axiom of universe, clearly, does not hold in this model. So the axiom of universe cannot be proved from ZFC.) We will not formally prove this, but instead explain why various constructions do not leave a chosen universe. In the following (lectures), we will choose a universe U and say that a set x is <u>small</u> if $x \in U$ and <u>large</u> if $x \notin U$.

Let U be a universe. We list a number of consequences of the definition.

L1 If $x \in U$ and $y \subset x$, then $y \in U$.

Pf $y \in \mathcal{P}(x) \in U$.

L2 If $x \in U$ and y is a quotient of x, then $y \in U$.

Pf $y \subset \mathcal{P}(x) \in U$.

L3 If $x \in U$, then $\{x\} \in U$.

Pf $\{x\} = \{x, x\} \in U$.

L4 If $x, y \in U$, then

$$(x, y) := \{\{x\}, \{x, y\}\} \in U.$$

L5 If $X, Y \in U$, then $X \times Y \in U$.

Pf For $x \in X$, we consider the function

$$Y \xrightarrow{g} U, \quad g(y) = (x, y)$$

to see that

$$\{x\} \times Y = \bigcup_{y \in Y} g(y) \in U.$$

Next, we consider the function

$$X \xrightarrow{f} U, \qquad f(x) = \{x\} \times Y$$

to see that

$$X \times Y = \bigcup_{x \in X} f(x) \in U. \qquad //$$

L6 If $I \in U$ and $x : I \to U$ is a function, then $\coprod_{i \in I} x(i) \in U$.

Pf For the disjoint union is a subset

$$\coprod_{i \in I} x(i) \subset \left(\bigcup_{i \in I} x(i) \right) \times I. \qquad //$$

L7 If $x, y \in U$, then every function $f : x \to y$ is an element of U.

Pf For $f = \Gamma_f \subset x \times y \in U$.

L8 If $x, y \in U$, then $\mathrm{Hom}(x, y) \in U$.

Pf For $\mathrm{Hom}(x, y) \subset \{x\} \times \{y\} \times \mathcal{P}(x \times y)$. //

L9 If $I \in U$ and $x : I \to U$ is a function, then $\prod_{i \in I} x(i) \in U$.

Pf For $\prod_{i \in I} x(i) \subset \mathrm{Hom}(I, \bigcup_{i \in I} x(i)) \in U$. //

A category $\mathcal{C}$, by definition, is a sextuple

$$\mathcal{C} = \Big(ob(\mathcal{C}),\ mor(\mathcal{C}),\ ob(\mathcal{C}) \xrightarrow{1} mor(\mathcal{C}),$$

$$mor(\mathcal{C}) \xrightarrow{s} ob(\mathcal{C}),\ mor(\mathcal{C}) \xrightarrow{t} ob(\mathcal{C}),$$

$$\underset{s \searrow\, ob(\mathcal{C}) \,\swarrow t}{mor(\mathcal{C}) \times mor(\mathcal{C})} \xrightarrow{\circ} mor(\mathcal{C}) \Big)$$

where $ob(\mathcal{C})$ and $mor(\mathcal{C})$ are sets and 1, s, t, and $\circ$ are maps that satisfy some well-known axioms. So a category $\mathcal{C}$, as every mathematical object, is a certain kind of set. By (L4) and (L7), we have

$$\mathcal{C} \in U \iff ob(\mathcal{C}),\ mor(\mathcal{C}) \in U.$$

In this case, we say that $\mathcal{C}$ is a small category w.r.t. U.

We define the category U-Set of small sets by

$$ob(U\text{-Set}) = U$$

$$mor(U\text{-Set}) = \bigcup_{(x,y) \in U \times U} \mathrm{Hom}(x,y)$$

with 1, s, t, and ∘ defined as usual. We note that the union defining mor(U-Set) is disjoint, since, by definition,

$$\mathrm{Hom}(x,y) \subset \{x\} \times \{y\} \times \mathcal{P}(x \times y).$$

We recall from (L8) that each of sets Hom(x, y) is small. Hence,

$$\mathrm{mor}(U\text{-Set}) \subset U,$$

but is not itself a small set. So U-Set is locally small: it has a large set of objects, but all Hom-sets are small.

The category U-Top of small top. spaces is defined similarly. Recall that a topological space is a pair (X, τ) with $\tau \subset \mathcal{P}(X)$. So

$$(X, \tau) \in U \iff X \in U.$$

Also,

$$\mathrm{Hom}_{\mathrm{Top}}((X,\tau),(X',\tau'))$$
$$\subset \mathrm{Hom}_{\mathrm{Set}}(X, X')$$

so U-Top again is a locally small category. Note that, counter-intuitively, we have

$$\mathrm{ob}(U\text{-}\mathrm{Top}) \subset \mathrm{ob}(U\text{-}\mathrm{Set}),$$

which does not mean that U-Top is a subcategory of U-Set. (Why not?)

Let V be a larger universe with $U \in V$, hence, $U \subset V$. Then we have a fully faithful functor

$$U\text{-}\mathrm{Set} \hookrightarrow V\text{-}\mathrm{Set}$$

given by the inclusion on objects.

Finally, we note that if $I \in U$ is a small category and

$$I \xrightarrow{X} U\text{-}\mathrm{Set}$$

is a functor, then

$$\lim_I X,\ \mathrm{colim}_I X \in U.$$

For $\lim_I X$ is a subset of $\prod_{i \in I} X(i)$ and $\mathrm{colim}_I X$ is a quotient of $\coprod_{i \in I} X(i)$.