We will prove the following result in a series of steps.

Prop Every finite diagram of schemes admits a limit.

It will suffice to show that the category of schemes has a terminal object and that fiber products exist. We recall the adjunction

$$\begin{tikzcd} \text{Schemes} \arrow[r, shift left, "\Gamma(-,-)"] & \text{Rings}^{op} \arrow[l, shift left, "\text{Spec}"] \end{tikzcd}$$

It shows that $\text{Spec}(\mathbb{Z})$ is a terminal object and that if $f: A \to C$ and $g: B \to C$ are ring homom., then

$$\begin{tikzcd} \text{Spec}(A \otimes_C B) \arrow[r, "in_1^*"] \arrow[d, "in_2^*"] & \text{Spec}(A) \arrow[d, "f^*"] \\ \text{Spec}(B) \arrow[r, "g^*"] & \text{Spec}(C) \end{tikzcd}$$

is a fiber products. It follows that fiber products of affine schemes exist. (A scheme is affine if it isomorphic to Spec(A) for some ring A.) We will use the following lemma.

<u>Lemma</u> If $(j,j^{\#}):(U,\mathcal{O}_U)\hookrightarrow(X,\mathcal{O}_X)$ is an open immersion of schemes and if $(Z,\mathcal{O}_Z)$ is any scheme, then the diagram

$$
\begin{tikzcd}
\mathrm{Hom}_{Sch}((Z,\mathcal{O}_Z),(U,\mathcal{O}_U)) \arrow[r] \arrow[d, "(j,j^{\#})_*"] & \mathrm{Hom}_{Set}(Z,U) \arrow[d, "j_*"] \\
\mathrm{Hom}_{Sch}((Z,\mathcal{O}_Z),(X,\mathcal{O}_X)) \arrow[r] & \mathrm{Hom}_{Set}(Z,X)
\end{tikzcd}
$$

is cartesian.

<u>Pf</u> Exercise in definition of subspace topology and open immersion. //

Universal property of fiber product:

$$
\begin{tikzcd}
Z \arrow[dr, dashed, "\exists! w"] \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "v"'] & & \\
& X\times_S Y \arrow[r, "p"] \arrow[d, "q"] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"] & S
\end{tikzcd}
$$

<u>Lemma</u> If $X\times_S Y$ exists and $j:U\hookrightarrow X$ is an open immersion, then $U\times_S Y$ exists.

Pf The previous lemma shows that

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "p|U"] \arrow[d, "g|p^{-1}(U)"'] & U \arrow[d, "f\circ j"] \\ Y \arrow[r, "g"] & S \end{tikzcd}$$

has the universal property of $U \times_S Y$. Here we consider $p^{-1}(U)$ as an open subscheme of $X \times_S Y$. //

Lemma Let $\{X_i\}_{i\in I}$ be an open covering of X and suppose that for all $i \in I$, $X_i \times_S Y$ exists. Then $X \times_S Y$ exists.

Pf For $\emptyset \neq T \subset I$ finite, set $X_T = \bigcap_{i\in T} X_i$. Then $T \longmapsto X_T$ is a strongly cocartesian punctured I-cubical diagram in which every morphism is an open immersion and X is the colimit of this diagram. The previous lemma shows that for all $\emptyset \neq T \subset I$ finite, $X_T \times_S Y$ exists and

$$T \longmapsto X_T \times_S Y$$

is a punctured I-cubical diagram in which every morphism is an open immersion and we claim that it is strongly cocart. Indeed,

$$\begin{tikzcd} X_T \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\text{Cocart.}"] & X_{T \setminus \{i\}} \arrow[d, hook] \\ X_{T \setminus \{j\}} \arrow[r, hook] & X_{T \setminus \{i,j\}} \end{tikzcd}$$

$$\Longleftrightarrow$$

$$\begin{tikzcd} X_T \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\text{Cart.}"] & X_{T \setminus \{i\}} \arrow[d, hook] \\ X_{T \setminus \{j\}} \arrow[r, hook] & X_{T \setminus \{i,j\}} \end{tikzcd}$$

$$\Longrightarrow$$

$$\begin{tikzcd} X_T \times_S Y \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\text{Cart.}"] & X_{T \setminus \{i\}} \times_S Y \arrow[d, hook] \\ X_{T \setminus \{j\}} \times_S Y \arrow[r, hook] & X_{T \setminus \{i,j\}} \times_S Y \end{tikzcd}$$

$$\Longleftrightarrow$$

$$\begin{tikzcd} X_T \times_S Y \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\text{Cocart.}"] & X_{T \setminus \{i\}} \times_S Y \arrow[d, hook] \\ X_{T \setminus \{j\}} \times_S Y \arrow[r, hook] & X_{T \setminus \{i,j\}} \times_S Y \end{tikzcd}$$

By the gluing lemma, the colimit of $T \longmapsto X_T \times_S Y$ exists. We proceed to show that

$$X \times_S Y = \operatorname*{colim}_T (X_T \times_S Y) .$$

So consider a diagram as on page 2

and let $Z_T = p^{-1}(Z)$. Then

$$
\begin{tikzcd}
Z \arrow[r, "w"] \arrow[d, equal] & X \times_S Y \arrow[d, equal] \\
\operatorname{colim}_T Z_T \arrow[r, "w"] & \operatorname{colim}_T X_T \times_S Y \\
Z_T \arrow[u, "\mathrm{in}_T"] \arrow[r, "\exists!\, w_T"] \arrow[ur, phantom, "\Uparrow"] & X_T \times_S Y \arrow[u, "\mathrm{in}_T"']
\end{tikzcd}
$$

The lemma at the top of page 2 shows that $\mathrm{in}_T \circ w_T$ is the unique morphism that makes the diagram

$$
\begin{tikzcd}
Z_T \arrow[dr] \arrow[drr, bend left, "u \circ \mathrm{in}_T"] \arrow[ddr, bend right, "v \circ \mathrm{in}_T"'] & & \\
& X \times_S Y \arrow[r, "p"] \arrow[d, "q"] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"] & S
\end{tikzcd}
$$

commute. By the universal property of the colimit, the map w is the unique map that makes the diagram on page 2 commute. This shows that $X \times_S Y$ is indeed the fiber product. //

<u>Lemma</u> If S is affine, then $X \times_S Y$ exists.

<u>Pf</u> Let $\{X_i\}_{i \in I}$ and $\{Y_j\}_{j \in J}$ be affine open covers of X and Y, respectively. As noted on page 1, $X_i \times_S Y_j$ exists for all $i \in I$ and $j \in J$. By the previous lemma, $X_i \times_S Y$ exists for all $i \in I$, and by another instance of the same lemma, $X \times_S Y$ exists. //

<u>Prop</u> $X \times_S Y$ exists.

<u>Pf</u> Let $\{S_i\}_{i \in I}$ be an affine open covering of S, and set $X_i = f^{-1}(X)$ and $Y_i = g^{-1}(Y)$. The previous lemma shows that $X_i \times_{S_i} Y_i$ exists for all $i \in I$, and the lemma at the top of page 2 shows that

$$X_i \times_{S_i} Y_i = X_i \times_S Y_i ,$$

since the two sides have the same universal property. For $\emptyset \neq T \subset I$ finite, let $X_T = \bigcap_{i \in T} X_i$ and $Y_T = \bigcap_{i \in T} Y_i$. The lemma at the bottom of page 2 shows that $X_T \times_S Y_T$ exists and the argument on page 4 shows that

$$T \longmapsto X_T \times_S Y_T$$

is a strongly cocartesian punctured I-cubical diagram in which every morphism is an open immersion. Finally, we argue as in the proof of the lemma on page 3 that

$$X \times_S Y := \operatorname*{colim}_T X_T \times_S Y_T$$

is a fiber product as desired. //

We often think of a morphism of schemes $f: X \to S$ as a family of schemes parametrized by S. If $s \in S$, then there is a canonical map of schemes

$$\operatorname{Spec} k(s) \xrightarrow{i_s} S$$

with $i_s^{\#}: \mathcal{O}_{S,s} \to k(s)$ the canonical projection. The fiber product

$$\begin{tikzcd} X_s \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} k(s) \arrow[r, "i_s"] & S \end{tikzcd}$$

is called the <u>fiber of f over</u> $s \in S$.

We say that a morphism of schemes $f: X \to S$ is <u>surjective</u> if the underlying map of ~~sets~~ is surjective.

<u>Prop</u> If $f: X \to S$ is surjective and if $g: S' \to S$ is any morphism, then the induced map $f': X \times_S S' \to S'$ is surjective.

<u>Pf</u> First, if k is a field, then a morphism of schemes

$$\operatorname{Spec} k \longrightarrow X$$

determines and is determined by a point $x \in X$ and a ring homomorphism $k(x) \hookrightarrow k$.

Now, let $s' \in S'$, let $s = g(s) \in S$, and choose $x \in X$ s.t. $f(x) = s$. Then

$$\begin{tikzcd}
\operatorname{Spec} k(s') \otimes_{k(s)} k(x) \arrow[r] \arrow[d] & \operatorname{Spec} k(x) \arrow[d] \\
\operatorname{Spec} k(s') \arrow[r] & \operatorname{Spec} k(s)
\end{tikzcd}$$

is cartesian. So choose any ring homomorphism from $k(s') \otimes_{k(s)} k(x)$ to a field k. Then, in the diagram,

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[rr] \arrow[dd] \arrow[dr, dashed, "\exists!"] & & \operatorname{Spec} k(x) \arrow[dr] \arrow[dd] & \\
& X \times_S S' \arrow[rr] \arrow[dd] & & X \arrow[dd] \\
\operatorname{Spec} k(s') \arrow[rr] \arrow[dr] & & \operatorname{Spec} k(s) \arrow[dr] & \\
& S' \arrow[rr] & & S
\end{tikzcd}$$

the back face commutes and the front face is cartesian; whence the morphism making the left-hand face commute exists. So $X \times_S S' \to S'$ is surjective. //

We refer to this result as

"surjective" = "universally surjective."

Ex Call a morphism $f: X \to S$ injective if the underlying map of sets is so. Then

"injective" $\neq$ "universally injective."

For let L/K be a finite Galois ext. of fields. Then

$$L \otimes_K L \xrightarrow{\sim} \prod_{\sigma \in G} L$$

$$x \otimes y \longmapsto (x\sigma(y))$$

so while $\operatorname{Spec} L \to \operatorname{Spec} K$ is injective,

$$\begin{array}{ccc} \coprod_{\sigma \in G} \operatorname{Spec} L & \xrightarrow{\sim} & \operatorname{Spec} L \times_{\operatorname{Spec} K} \operatorname{Spec} L \\ \downarrow \Sigma \sigma^* & & \downarrow \\ \operatorname{Spec} L & = & \operatorname{Spec} L \end{array}$$

is not."

Def A universally injective morphism is called radical.

Ex (1) If $x \in X$, then the morphism

$$\operatorname{Spec} k(x) \xrightarrow{i_x} X$$

is radical. In fact, it is a monomorphism in the category of ringed spaces.

(2) If L/K is a purely inseparable extension of fields, then

$$\operatorname{Spec} L \longrightarrow \operatorname{Spec} K$$

is radical.

Prop Let $f: X \to S$ be a morphism of schemes and let $s \in S$. Then the cartesian square of schemes

$$\begin{tikzcd} X_s \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} k(s) \arrow[r, "i_s"] & S \end{tikzcd}$$

induces a cartesian square of underlying sets.

Pf The statement is that the map of sets $X_s \to f^{-1}(s)$ is a bijection. The map is injective, since i_s is radicial, and it is surjective by (the proof of) the proposition on page 8. //

Rmk It is proved in EGA I 3.6.1 that $X_s \to f^{-1}(s)$ is in fact a homeomorphism, if $f^{-1}(s) \subset X$ is given the subspace topology. //

In general, however, the underlying set of $X \times_S Y$ is not the fiber product of the underlying sets of X and Y over S. There is a different notion of point, however.

We say that a morphism of schemes

$$T \xrightarrow{g} X$$

is a T-valued point in X and write $X(T)$ for the set of T-valued points in X. If $T = \mathrm{Spec}(A)$, we write $X(A)$ for $X(T)$ and say that this is the set of A-valued points in X. A point $x \in X$ in the usual sense determines the $k(x)$-valued point $i_x : \mathrm{Spec}\, k(x) \to X$. Now, by the universal property of fiber product, the canonical map

$$(X \times_S Y)(T) \to X(T) \times_{S(T)} Y(T)$$

is a bijection.

Ex Suppose

$$X = \mathrm{Spec}(\mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_m)).$$

Then $X(A)$ is canonically bijective to the set of solutions $(a_1, \ldots, a_n)$ in A to the systems of eq.

$$f_i(x_1, \ldots, x_n) = 0 \qquad (1 \leq i \leq m). \;//$$

It is also helpful to think of $X(S)$ as the set of isomorphism classes of some kind of structure. For example, (in words to be defined later) define an <u>elliptic curve over</u> S to be a pair of morphisms

$$\begin{array}{c} E \\ f \downarrow \uparrow 0 \\ S \end{array}$$

s.t. $f \circ 0 = id_S$, f is smooth and proper of relative dimension 1, and $R^1 f_* \mathcal{O}_E$ is a locally free $\mathcal{O}_S$-mod. of rank 1. We could hope that there exists a universal elliptic curve

$$\begin{array}{c} \mathcal{E} \\ f \downarrow \uparrow 0 \\ \mathcal{M} \end{array}$$

such that the map

$$\mathcal{M}(S) \longrightarrow \left\{ \begin{array}{l} \text{isom. cl. of ell.} \\ \text{curves over } S \end{array} \right\}$$

$$(S \xrightarrow{g} \mathcal{M}) \longmapsto \text{cl. of } \mathcal{E} \times_{\mathcal{M}} S \underset{f}{\overset{0}{\rightleftarrows}} S$$

is a bijection. It turns out that such a (fine moduli) scheme M does not exist for the following reason. A ring homomorphism $f: A \to B$ is called <u>faithfully flat</u> if a sequence of A-modules

$$M' \longrightarrow M \longrightarrow M''$$

is exact if and only if the induced sequence of B-modules

$$M' \otimes_A B \longrightarrow M \otimes_A B \longrightarrow M'' \otimes_A B$$

is exact. This is equivalent to f being flat and $f^*: \operatorname{Spec} B \to \operatorname{Spec} A$ being surjective. The following is (a special case of) Lemma 31.9.3 in the Stacks Project.

<u>Thm</u> If $f: A \to B$ is a faithfully flat ring homomorphism, then the diagram of schemes

$$\operatorname{Spec} B \times_{\operatorname{Spec} A} \operatorname{Spec} B \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} \operatorname{Spec} B \xrightarrow{f} \operatorname{Spec} A$$

is a coequalizer. //

It follows that, in this situation, if X is any scheme, then

$$X(A) \xrightarrow{f^*} X(B) \underset{pr_2^*}{\overset{pr_1^*}{\rightrightarrows}} X(B) \times_{X(A)} X(B)$$

is an equalizer diagram of sets. In particular, $f^*: X(A) \to X(B)$ is injective.

Now, in the case of elliptic curves, it is possible to find an extension of fields L/K (so $K \to L$ is faithfully flat) and elliptic curves E, E' over K that are not isomorphic over K but such that $E \times_{\mathrm{Spec}\,K} \mathrm{Spec}\,L$ and $E' \times_{\mathrm{Spec}\,K} \mathrm{Spec}\,L$ are isomorphic as elliptic curves over L. (See Silverman's book, Section 10.2, for a concrete example.) So a fine moduli scheme of elliptic curves cannot exist. However, for elliptic curves with (a suitable notion of) level structure, a fine moduli scheme and a universal elliptic curve with level structure over it does exist.