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The (large) category $X^\sim$ of sheaves of (small) sets on a (small) space X has all (small) limits and colimits. Specifically, if

$$I \xrightarrow{F} X^\sim$$

is a (small) diagram in $X^\sim$, then

$$(\lim_I F_i)(U) = \lim_I F_i(U)$$

$$(\operatorname{colim}_I F_i)(U)$$
$$= a(V \mapsto \operatorname{colim}_I F_i(V))(U).$$

Let $\mathcal{O}_X \in X^\sim$ be a ring object, i.e. $(X, \mathcal{O}_X)$ is a ringed space. The category $\mathrm{Mod}\,\mathcal{O}_X$ of $\mathcal{O}_X$-modules again has all (small) limits and colimit; in particular, it has all (small) products and sums and kernels and cokernels. Note that taking infinite sums and taking cokernels involve sheafification, e.g.

$$(\bigoplus_{i \in I} M_i)(U) = a(V \mapsto \bigoplus_{i \in I} M_i(V))(U).$$

Def Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module M is <u>quasicoherent</u> if there exists an open covering

$$\{ U_\alpha \xrightarrow{j_\alpha} X \}_{\alpha \in A}$$

s.t. for all $\alpha \in A$, there exists a presentation

$$\bigoplus_{i \in I_{1,\alpha}} j_\alpha^* \mathcal{O}_X \to \bigoplus_{i \in I_{0,\alpha}} j_\alpha^* \mathcal{O}_X \to j_\alpha^* M \to 0$$

of the $j_\alpha^* \mathcal{O}_X$-module $j_\alpha^* M$. //

If $j : U \to X$ is the canonical inclusion of an open subset, we write

$$M|_U = j^* M;$$

compare Recitation 2.

If $(X, \mathcal{O}_X) = (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ and if M is an A-module, then we define $\tilde{M}$ to be the $\mathcal{O}_X$-module

$$\tilde{M} = a\left(U \mapsto \begin{cases} M_f & \text{if } U = D(f) \\ 0 & \text{otherwise} \end{cases} \right)$$

We have $\tilde{A} = \mathcal{O}_X$ and, as in That case, there are canonical isomorph.

$$\Gamma(D(f), \tilde{M}) \xleftarrow{\sim} M_f$$

$$\tilde{M}_x \xrightarrow{\sim} M_{\mathfrak{p}_x}$$

<u>Lemma</u> Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$. There is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, N) \xrightarrow{\sim} \mathrm{Hom}_A(M, \Gamma(X, N))$$

$$f \longmapsto (M \xrightarrow{\sim} \Gamma(X, \tilde{M}) \xrightarrow{f_X} \Gamma(X, N))$$

<u>Pf</u> Inverse map: If $g : M \to \Gamma(X, N)$ is A-linear, then there is a unique map of A_f-modules that makes the diagram

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d] & \Gamma(X, N) \arrow[d, "\rho^X_{D(f)}"] \\ M_f \arrow[r, dashed, "\exists!"] & \Gamma(D(f), N) \end{tikzcd}$$

commute. //

Let Mod_A and $\mathrm{Mod}_{\mathcal{O}_X}$ be the (locally small, large) categories of A-modules and $\mathcal{O}_X$-modules, respectively. The lemma shows that

$$\mathrm{Mod}_A \underset{\Gamma(X,-)}{\overset{(-)^\sim}{\rightleftarrows}} \mathrm{Mod}_{\mathcal{O}_X}$$

is an adjunction. We note that the right adjoint preserves finite limits, because sheafification does so, and that the unit map

$$M \xrightarrow{\eta} \Gamma(X, \tilde{M})$$

is an isomorphism. Hence, the right adjoint is fully faithful, i.e.

$$\begin{array}{ccc} \mathrm{Hom}_A(M,N) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M},\tilde{N}) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ f & \longmapsto & \tilde{f} \end{array}$$

We will now prove that the quasi-coherent $\mathcal{O}_X$-modules precisely are the $\mathcal{O}_X$-modules that are isomorphic to $\mathcal{O}_X$-modules in the image of the functor

$$\mathrm{Mod}_A \xrightarrow{(-)^\sim} \mathrm{Mod}_{\mathcal{O}_X}.$$

<u>Thm</u> Let $(X, \mathcal{O}_X)$ be a scheme and let M be an $\mathcal{O}_X$-module. The following are equivalent:

(1) M is quasicoherent;

(2) for all $U \subset V \subset X$ affine open,

$$\Gamma(U, \mathcal{O}_X) \otimes_{\Gamma(V, \mathcal{O}_X)} \Gamma(V, M) \xrightarrow{\sim} \Gamma(U, M);$$

(3) for all $U \subset M$ affine open,

$$\widetilde{\Gamma(U, M)} \xrightarrow[\sim]{\varepsilon} M|_U;$$

(4) there exists an affine open covering $\{U_\alpha\}_{\alpha \in A}$ of X s.t. for all $\alpha \in A$,

$$\widetilde{\Gamma(U_\alpha, M)} \xrightarrow[\sim]{\varepsilon} M|_{U_\alpha}.$$

<u>Pf</u> (1) $\Rightarrow$ (4): We choose an affine open covering $\{U_\alpha\}_{\alpha \in A}$ of X s.t. for every $\alpha \in A$, there exists a presentation

$$\bigoplus_{i \in I_{1,\alpha}} \mathcal{O}_X|_{U_\alpha} \xrightarrow{h_\alpha} \bigoplus_{i \in I_{0,\alpha}} \mathcal{O}_X|_{U_\alpha} \to M|_{U_\alpha} \to 0.$$

Since $\widetilde{(-)}$ preserves colimits,

$$\bigoplus_{i\in I} \mathcal{O}_X|U_\alpha \xleftarrow{\sim} \bigoplus_{i\in I} \widetilde{\Gamma(U_\alpha, \mathcal{O}_X)}$$
$$\xrightarrow{\sim} \Big(\bigoplus_{i\in I} \Gamma(U_\alpha, \mathcal{O}_X)\Big)^{\sim},$$

and since $\widetilde{(-)}$ is fully faithful, the map h_α is induced by a map of $\Gamma(U_\alpha, \mathcal{O}_X)$-modules

$$\bigoplus_{i\in I_{1,\alpha}} \Gamma(U_\alpha, \mathcal{O}_X) \xrightarrow{h'_\alpha} \bigoplus_{i\in I_{0,\alpha}} \Gamma(U_\alpha, \mathcal{O}_X).$$

Let N_α be a cokernel of h'_α. Since $\widetilde{(-)}$ preserves colimits, we conclude from the isom. at the top of the page that the canonical map

$$\widetilde{N}_\alpha \longrightarrow M|U_\alpha$$

is an isomorphism. Since the unit η of the adjunction $(\widetilde{(-)}, \Gamma(X,-))$ is an isomorphism, we conclude that

$$\Gamma(U_\alpha, M)^{\sim} \xrightarrow[\sim]{\varepsilon} M|U_\alpha$$

as desired. (Exercise in adjunctions: Fill in the details of this last argument.)

(4) ⇒ (3): Let $U \subset X$ be an affine open. We choose, using (4), an affine open covering $\{U_\alpha\}_{\alpha \in A}$ of U s.t. for all $\alpha \in A$,

$$\Gamma(U_\alpha, M)^\sim \xrightarrow[\sim]{\varepsilon} M|U_\alpha .$$

Since U is affine, we may assume that $U_\alpha = U_{g_\alpha}$ with $g_\alpha \in \Gamma(U, \mathcal{O}_X)$ is a distinguished. In particular, $U_\alpha \cap U_\beta$ again is affine. Since U is quasicompact, we may further assume that A is finite. Let

$$U_\alpha \xrightarrow{j_\alpha} U \,, \quad U_\alpha \cap U_\beta \xrightarrow{j_{\alpha\beta}} U$$

be the canonical inclusions. Since M is a sheaf, the diagram of $\mathcal{O}_X|_U$-modules

$$M|_U \to \prod_\alpha j_{\alpha *}(M|_{U_\alpha}) \rightrightarrows \prod_{\alpha,\beta} j_{\alpha\beta *}(M|_{U_\alpha \cap U_\beta})$$

is an equalizer. Since $\Gamma(U, -)$ preserves limits and $\widetilde{(-)}$ preserves finite limits, it suffices to show that the counit maps

$$\Gamma(U, j_{\alpha *}(M|U_\alpha))^\sim \xrightarrow{\varepsilon} j_{\alpha *}(M|U_\alpha)$$

and

$$\Gamma(U, j_{\alpha\beta *}(M|U_\alpha \cap U_\beta))^\sim \xrightarrow{\varepsilon} j_{\alpha\beta *}(M|U_\alpha \cap U_\beta)$$

are isom. We factor the first of these as

$$\Gamma(U, j_{\alpha *}(M|_{U_\alpha}))^\sim \longrightarrow j_{\alpha *}(\Gamma(U_\alpha, M|U_\alpha)^\sim) \xrightarrow[\sim]{j_{\alpha *}\varepsilon} j_{\alpha *}(M|U_\alpha)$$

where the last map is an isomorphism by assumption. To show that the first map is an isom., it will suffice to consider the induced map of sections over $D(f)$, $f \in \Gamma(U, \mathcal{O}_X)$. We have canonical isomorphisms

$$\Gamma(D(f), \Gamma(U, j_{\alpha *}(M|U_\alpha))^\sim) \xleftarrow{\sim} \Gamma(U, j_{\alpha *}(M|U_\alpha))_f = \Gamma(U_\alpha, M|U_\alpha)_f$$

and

$$\Gamma(D(f), j_{\alpha *}(\Gamma(U_\alpha, M|U_\alpha)^\sim))$$

$$= \Gamma(D(fg_\alpha), \Gamma(U_\alpha, M|U_\alpha)^\sim)$$

$$\xleftarrow{\sim} \Gamma(U_\alpha, M|U_\alpha)_{fg_\alpha},$$

and the map in question is the localization map

$$\Gamma(U_\alpha, M|U_\alpha)_f \to \Gamma(U_\alpha, M|U_\alpha)_{fg_\alpha}.$$

It is an isomorphism since g_α is a unit in $\Gamma(U_\alpha, \mathcal{O}_X)$.

(3) $\Rightarrow$ (2): We choose a presentation of the $\Gamma(V, \mathcal{O}_X)$-module $\Gamma(V, M)$,

$$\bigoplus_{i \in I_1} \Gamma(V, \mathcal{O}_X) \to \bigoplus_{i \in I_0} \Gamma(V, \mathcal{O}_X) \to \Gamma(V, M) \to 0$$

Since, by assumption, the counit map

$$\Gamma(V, M)^\sim \xrightarrow{\varepsilon} M|_V$$

is an isomorphism, and since $(-)^\sim$ preserves colimits, we obtain a presentation

$$\bigoplus_{i \in I_1} \mathcal{O}_X|_V \to \bigoplus_{i \in I_0} \mathcal{O}_X|_V \to M|_V \to 0.$$

Now, the restriction to U functor $(-)|_U$ also preserves colimits (having direct image as a right adjoint), so we obtain the induced presentation in the top row of the following commutative diagram.

$$\begin{tikzcd}
\bigoplus_{i\in I_1} \mathcal{O}_X|_U \arrow[r] & \bigoplus_{i\in I_0} \mathcal{O}_X|_U \arrow[r] & M|_U \arrow[r] & 0 \\
\left(\bigoplus_{i\in I_1} \Gamma(U,\mathcal{O}_X)\right)^{\sim} \arrow[u, "\sim"'] \arrow[r] & \left(\bigoplus_{i\in I_0} \Gamma(U,\mathcal{O}_X)\right)^{\sim} \arrow[u, "\sim"'] \arrow[r] & \Gamma(U,M)^{\sim} \arrow[u, "\sim"'] \arrow[r] & 0
\end{tikzcd}$$

The left-hand and middle vertical maps are isomorphisms, since on the affine scheme U,

$$\Gamma(U,\mathcal{O}_X)^{\sim} \xrightarrow[\sim]{\varepsilon} \mathcal{O}_X|_U ,$$

and since $(-)^{\sim}$ preserves colimits, and the right-hand vertical map is an isomorphism by assumption. Hence the bottom row is exact, and since $(-)^{\sim}$ is fully faithful, this row is the image by $(-)^{\sim}$ of an exact sequence of $\Gamma(U,\mathcal{O}_X)$-modules

$$\bigoplus_{i\in I_1} \Gamma(U,\mathcal{O}_X) \to \bigoplus_{i\in I_1} \Gamma(U,\mathcal{O}_X) \to \Gamma(U,M) \to 0.$$

Comparing this sequence to the exact sequence obtained by applying the right exact functor

$$\Gamma(U, \mathcal{O}_X) \underset{\Gamma(V, \mathcal{O}_X)}{\otimes} -$$

to the presentation at the bottom of page 9, we conclude that the canonical map

$$\Gamma(U, \mathcal{O}_X) \underset{\Gamma(V, \mathcal{O}_X)}{\otimes} \Gamma(V, M) \longrightarrow \Gamma(U, M)$$

is an isomorphism.

(2) $\Rightarrow$ (1): It suffices to show that for every affine open $U \subset X$, the $\mathcal{O}_X|_U$-module $M|_U$ has a presentation. A presentation

$$\bigoplus_{i \in I_1} \Gamma(U, \mathcal{O}_X) \to \bigoplus_{i \in I_0} \Gamma(U, \mathcal{O}_X) \to \Gamma(U, M) \to 0$$

of the $\Gamma(U, \mathcal{O}_X)$-module $\Gamma(U, M)$ gives rise to a presentation

$$\bigoplus_{i \in I_1} \mathcal{O}_X|_U \to \bigoplus_{i \in I_0} \mathcal{O}_X|_U \to \Gamma(U, M)^{\sim} \to 0$$

of the $\mathcal{O}_X|_U$-module $\Gamma(U,M)^\sim$, since $(-)^\sim$ preserves colimits, and since on the affine scheme U,

$$\Gamma(U,\mathcal{O}_X)^\sim \xrightarrow[\sim]{\varepsilon} \mathcal{O}_X|_U .$$

So it will suffice to show that

$$\Gamma(U,M)^\sim \xrightarrow{\varepsilon} M|_U$$

is an isomorphism (i.e. (2) $\Rightarrow$ (3)). Now, for all $f \in \Gamma(U,\mathcal{O}_X)$, we have a commutative diagram

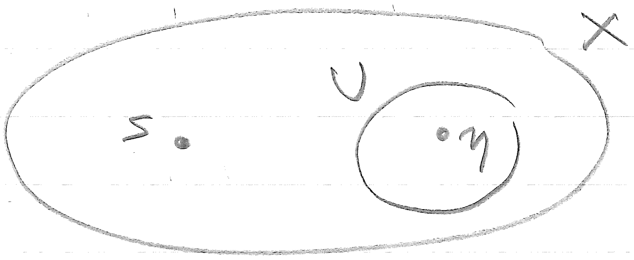
$$\begin{tikzcd}
\Gamma(U_f, \Gamma(U,M)^\sim) \arrow[rr] & & \Gamma(U_f, M|_U) \\
& \Gamma(U_f,\mathcal{O}_X) \otimes_{\Gamma(U,\mathcal{O}_X)} \Gamma(U,M) \arrow[ul, "\sim"] \arrow[ur, "\sim"'] &
\end{tikzcd}$$

in which the left-hand map is an isomorphism by the definition of $\Gamma(U,M)^\sim$ and the right-hand is an isomorphism by (2). Hence, the top horizontal is an isomorphism as desired. //

Ex Let V be a discrete valuation ring with maximal ideal m and quotient field K. We have

$$\mathrm{Spec}(V) = \{ s, \eta \}$$

with $\mathfrak{p}_s = m$ and $\mathfrak{p}_\eta = (0)$. The point s is closed and is called the special point, and the point η is open and is called the generic point, since the closure of $\{\eta\} \subset \mathrm{Spec}(V)$ is the whole space.



A sheaf M on $\mathrm{Spec}(V)$ consists of two sets and a map:

$$\begin{array}{ccc} M_0 & = & \Gamma(X, M) \\ \downarrow f & & \downarrow \rho^X_U \\ M_1 & = & \Gamma(U, M) \end{array}$$

An $\mathcal{O}_X$-module structure on M consists of a V-module structure on M_0 and a K-vector space structure on M_1 s.t. the map f is V-linear. Finally, this $\mathcal{O}_X$-module is quasicoherent if and only if the induced map of K-vector spaces

$$K \underset{V}{\otimes} M_0 \longrightarrow M_1$$

is an isomorphism. "