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Recall that a morphism

$$(U, \mathcal{O}_U) \xhookrightarrow{j} (X, \mathcal{O}_X)$$

is an open immersion if $j : U \to X$ is a homeomorphism onto an open subset of X and if $j^\# : j^* \mathcal{O}_X \to \mathcal{O}_U$ is an isomorphism. We define

$$(Z, \mathcal{O}_Z) \xhookrightarrow{i} (X, \mathcal{O}_X)$$

to be a closed immersion if $i : Z \to X$ is a homeomorphism onto a closed subset of X, if $i^\# : \mathcal{O}_X \to i_* \mathcal{O}_Z$ is surjective, and if the kernel

$$0 \to I \to \mathcal{O}_X \xrightarrow{i^\#} i_* \mathcal{O}_Z \to 0$$

is a quasicoherent ideal in $\mathcal{O}_X$. If $I \subset \mathcal{O}_X$ is any ideal, then

$$\operatorname{supp}(\mathcal{O}_X / I)$$

$$= \{ x \in X \mid (\mathcal{O}_X / I)_x \neq 0 \}$$

is a closed subset $Z \subset X$. (For let $\bar{1} \in \Gamma(X, \mathcal{O}_X / I)$ be the identity global section. Then $\operatorname{supp}(\mathcal{O}_X / Z)$ is equal to $\operatorname{supp}(\bar{1})$ and the support of a global section of a sheaf of abelian groups is always closed, by the definition of stalks.) Let $\mathcal{O}_Z = i^*(\mathcal{O}_X / I)$ s.t.

$$0 \to I \to \mathcal{O}_X \xrightarrow{\eta} i_* \mathcal{O}_Z \to 0$$

is exact. Then the ringed space $(Z, \mathcal{O}_Z)$ is a scheme if and only if $I \subset \mathcal{O}_X$ is quasicoherent, whence the definition; see EGA I.4. So closed subschemes of $(X, \mathcal{O}_X)$ are in canonical 1-1 correspondence with quasicoherent ideals of $\mathcal{O}_X$.

__Ex__ The closed subschemes of an affine scheme $(X, \mathcal{O}_X)$ are in 1-1 correspondence with ideals of the ring $\Gamma(X, \mathcal{O}_X)$. //

A morphism of schemes

$$(Y, \mathcal{O}_Y) \xrightarrow{f} (X, \mathcal{O}_X)$$

is an __immersion__ if it can be written as the composition

$$(Y, \mathcal{O}_Y) \xhookrightarrow{i} (U, \mathcal{O}_U) \xhookrightarrow{j} (X, \mathcal{O}_X)$$

of a closed immersion i and an open immersion j.

Prop Let $(X, \mathcal{O}_X)$ be a scheme. There is a unique quasicoherent ideal $\mathrm{nil}(\mathcal{O}_X) \subset \mathcal{O}_X$ s.t. for all $x \in X$,

$$\mathrm{nil}(\mathcal{O}_X)_x = \mathrm{nil}(\mathcal{O}_{X,x})$$

Pf We may assume that $X = \mathrm{Spec}(A)$ is affine. Define

$$\mathrm{nil}(\mathcal{O}_X) = \widetilde{\mathrm{nil}(A)}.$$

The inclusion

$$\mathrm{nil}(\mathcal{O}_X)_x \subset \mathrm{nil}(\mathcal{O}_{X,x})$$

is clear, so we proceed to show the opposite inclusion. Let $a/s \in \mathcal{O}_{X,x}$ and suppose $(a/s)^n = 0 \in \mathcal{O}_{X,x}$. This means that there exists $t \in \mathfrak{p}_x$ s.t. $ta^n = 0 \in A$. But then also $(ta)^n = 0 \in A$, so

$$a/s = ta/ts \in \mathrm{nil}(\mathcal{O}_X)_x$$

as desired. //

The closed subscheme

$$(X_{red}, \mathcal{O}_{X_{red}}) \xrightarrow{i} (X, \mathcal{O}_X)$$

define by the quasicoherent ideal $nil(\mathcal{O}_X) \subset \mathcal{O}_X$ is called the <u>reduced scheme</u> of X. Since $1 \in \mathcal{O}_{X,x}$ is never nilpotent, the space X_{red} is equal to X.

<u>Lemma</u> Let $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ be a morphism of schemes and suppose there exists a family $\{U_i\}_{i \in I}$ of open subsets of X such that $\{f^{-1}(U_i)\}_{i \in I}$ covers Y and such that for all $i \in I$,

$$(f^{-1}(U_i), \mathcal{O}_Y|f^{-1}(U_i)) \xrightarrow{f} (U_i, \mathcal{O}_X|U_i)$$

is a closed immersion. Then f is an immersion. If, in addition, the family $\{U_i\}_{i \in I}$ covers X, then f is a closed immersion.

<u>Pf</u> Exercise; or see EGA I.4.1.5. //

<u>Prop</u> If $f: X \to S$ is any morphism of schemes, then the diagonal

$$X \xrightarrow{\Delta_{X/S}} X \times_S X$$

is an immersion.

<u>Pf</u> If $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$ are affine, then

$$\begin{array}{ccc} X & \xrightarrow{\Delta_{X/S}} & X \times_S X \\ \| & & \| \\ \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(A \otimes_R A) \end{array}$$

is the closed immersion determined by the kernel $I \subset A \otimes_R A$ of the multiplication $\mu : A \otimes_R A \to A$.

In general, we choose two families $\{V_i\}_{i \in I}$ and $\{W_i\}_{i \in I}$ of affine open subsets of X and S, resp., s.t. for all $i \in I$, $f(V_i) \subset W_i$. Then $V_i \times_{W_i} V_i \subset X \times_S X$ is affine open and

$$\Delta_{X/S}^{-1}(V_i \times_{W_i} V_i) = V_i \subset X.$$

Hence, the lemma on page 4 shows that $\Delta_{X/S}$ is an immersion. //

<u>Def</u> A morphism of schemes $f: X \to S$ is <u>separated</u> if $\Delta_{X/S}$ is a closed immersion.

It follows from the (proof of) the proposition on page 5 that $f: X \to S$ is separated if and only if the diagonal $\Delta_{X/S}(X) \subset X \times_S X$ is a closed subset. We also saw that every morphism between affine schemes is separated. A scheme X is defined to be <u>separated</u> if

$$X \longrightarrow \mathrm{Spec}(\mathbb{Z})$$

is a separated morphism, i.e. if $\Delta_X : X \to X \times X$ is a closed immersion.

<u>Ex</u> Let X be the affine line over $\mathrm{Spec}(k)$ with the origin doubled:

$$\begin{tikzcd}
\mathrm{Spec}(k[t,t^{-1}]) \arrow[r, "t \leftarrow\!\shortmid x"] \arrow[d, "y \mapsto t"'] \arrow[dr, phantom, "\text{Cocart.}"] & \mathrm{Spec}(k[x]) \arrow[d] \\
\mathrm{Spec}(k[y]) \arrow[r] & X
\end{tikzcd}$$

Then $X \longrightarrow \mathrm{Spec}(k)$ is not separated. //

Let $i : Z \hookrightarrow X$ be the inclusion of a closed subspace. We recall (e.g. from recitation 2) that the adjunction

$$\begin{tikzcd} Z^{\sim} \arrow[r, shift right, "i_*"'] & X^{\sim} \arrow[l, shift right, "i^*"'] \end{tikzcd}$$

induces an adjoint equivalence of categories between $Z^{\sim}$ and the full subcategory of $X^{\sim}$ of sheaves on X supported on Z.

We now consider an immersion

$$\begin{tikzcd} (Y, \mathcal{O}_Y) \arrow[rr, "f"] \arrow[dr, hook, "i"'] & & (X, \mathcal{O}_X) \\ & (U, \mathcal{O}_U) \arrow[ur, hook, "j"'] & \end{tikzcd}$$

of schemes and define $\mathcal{J}_f \subset f^*\mathcal{O}_X$ to be the kernel

$$0 \to \mathcal{J}_f \to f^*\mathcal{O}_X \xrightarrow{f^\#} \mathcal{O}_Y \to 0$$

The locally ringed space

$$(Y^{(n)}, \mathcal{O}_{Y^{(n)}})$$

$$= (Y, f^*(\mathcal{O}_X)/\mathcal{J}_f^{n+1})$$

is a scheme. Indeed, let $I_f \subset \mathcal{O}_U$ be the quasicoherent ideal that defines $Y \subset U$, so $J_f = i^*(I_f)$. Then also

$$I_f^{n+1} = I_f \otimes_{\mathcal{O}_U} \cdots \otimes_{\mathcal{O}_U} I_f \quad (n+1 \text{ factors})$$

is a quasicoherent ideal of $\mathcal{O}_U$ and defines $Y^{(n)} \subset U$. The scheme $(Y^{(n)}, \mathcal{O}_{Y^{(n)}})$ is called the n'th infinitesimal neighborhood of Y in X. We have compatible factorizations

$$\begin{tikzcd} (Y, \mathcal{O}_Y) \arrow[rr, "f"] \arrow[dr, "i^{(n)}"'] & & (X, \mathcal{O}_X) \\ & (Y^{(n)}, \mathcal{O}_{Y^{(n)}}) \arrow[ur, "f^{(n)}"'] & \end{tikzcd}$$

with $i^{(n)}$ a closed immersion and $f^{(n)}$ an immersion. Note that the map of spaces $i^{(n)}: Y \to Y^{(n)}$ is the identity map. The quasi-coherent ideal

$$N_{Y/X} = J_f / J_f^2 \subset \mathcal{O}_{Y^{(1)}}$$

that defines $Y = Y^{(0)}$ in $Y^{(1)}$ is a quasicoherent $\mathcal{O}_Y$-module and is called the <u>conormal sheaf</u> of Y in X.

If $f : X \to S$ is any morphism of schemes, then $\Delta_{X/S} : X \to X \times_S X$ is an immersion. In this case, the conormal sheaf is written

$$\Omega^1_{X/S} = \mathcal{N}_{X/X\times_S X}$$

and is called the $\mathcal{O}_X$-module of <u>Kähler differentials</u> of X/S. It is a quasicoherent $\mathcal{O}_X$-module.