We consider the $\mathcal{O}_X$-module $\Omega^1_{X/S}$ in more detail. First, recall that if $f: R \to A$ is a ring homomorphism, then an R-linear <u>derivation</u> from A to an A-module M is an R-linear map

$$A \xrightarrow{D} M$$

s.t. for all $a, b \in A$, Leibniz rule

$$D(ab) = aD(b) + bD(a)$$

holds. We write $\mathrm{Der}_R(A, M)$ for the A-module of R-linear derivations from A to M. ~~Define~~

$$\begin{array}{ccccccccc} 0 & \to & I/I^2 & \to & A\otimes_R A/I^2 & \xrightarrow{\mu} & A & \to & 0 \\ & & \| & & \| & & \| & & \\ 0 & \to & \Omega^1_{A/R} & \to & P^1_{A/R} & \xrightarrow{\varepsilon} & A & \to & 0 \end{array}$$

where $I \subset A\otimes_R A$ is the kernel of the multiplication. It is an exact sequence of A-modules, where we consider $\Omega^1_{A/R}$ and $P^1_{A/R}$ as A-modules via multiplication in the left-hand tensor factor A. The map ε has the unique A-linear section $\eta_L : A \to P^1_{A/R}$ defined by $\eta_L(a) = a\otimes 1 + I^2$. The R-linear map

$$A \xrightarrow{d} \Omega^1_{A/R}$$

$$a \longmapsto 1\otimes a - a\otimes 1 + I^2$$

is a derivation. Indeed,

$$\begin{aligned} 0 &= (1\otimes a - a\otimes 1).(1\otimes b - b\otimes 1) \\ &= 1\otimes ab - b\otimes a - a\otimes b + ab\otimes 1 \\ &= 1\otimes ab - ab\otimes 1 - a\otimes 1(1\otimes b - b\otimes 1) \\ &\quad - b\otimes 1(1\otimes a - a\otimes 1). \end{aligned}$$

Moreover, the image of d generates $\Omega^1_{A/R}$ as an A-module. For if $\sum a_i\otimes b_i$ is any element of I, then

$$\sum a_i\otimes b_i = \sum a_i\otimes 1(1\otimes b_i - b_i\otimes 1).$$

It follows that if $h : \Omega^1_{A/R} \to M$ is an A-linear map, then

$$A \xrightarrow{d} \Omega^1_{A/R} \xrightarrow{h} M$$

is an R-linear derivation $D : A \to M$

and that h is determined by D.

Prop If M is an A-module, then the A-linear map

$$\mathrm{Hom}_A(\Omega^1_{A/R}, M) \longrightarrow \mathrm{Der}_R(A, M)$$

$$h \longmapsto h \circ d$$

is an isomorphism.

Pf We have already argued that the map is well-defined and injective, so only surjectivity is at issue. Let $D: A \to M$ be an R-linear derivation. It induces the A-linear map

$$A \otimes_R A \xrightarrow{\tilde{D}} M$$

$$a \otimes b \longmapsto a D(b)$$

and the calculation

$$(1 \otimes a - a \otimes 1)(1 \otimes b - b \otimes 1)$$

$$= 1 \otimes ab - b \otimes a - a \otimes b + ab \otimes 1$$

$$\longmapsto D(ab) - bD(a) - aD(b) + ab\, D(1)$$

$$= 0$$

shows that it factors through the A-linear map

$$P^1_{A/R} \xrightarrow{\bar{D}} M$$

$$a \otimes b + I^2 \longmapsto aD(b)$$

The restriction of $\bar{D}$ to $\Omega^1_{A/R}$ is an A-linear map

$$\Omega^1_{A/R} \xrightarrow{h} M$$

and, by definition,

$$(h \circ d)(a) = h(1 \otimes a - a \otimes 1 + I^2)$$

$$= D(a) - aD(1) = D(a),$$

which proves surjectivity. //

Rmk We note that $-d : A \to \Omega^1_{A/R}$ also is a universal R-linear derivation. //

We next let $f : X \to S$ be a morphism of schemes and recall from last time that the diagonal

$$X \xrightarrow{\Delta_{X/S}} X \times_S X$$

is an immersion and that $\Omega^1_{X/S}$ is defined to be the conormal sheaf. So we have an exact sequence of $\mathcal{O}_X$-modules

$$\begin{tikzcd}
0 \arrow[r] & J/J^2 \arrow[r] \arrow[d, equal] & \Delta^*(\mathcal{O}_{X\times_S X})/J^2 \arrow[r, "\Delta^\#"] \arrow[d, equal] & \mathcal{O}_X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega^1_{X/S} \arrow[r] & \mathcal{P}^1_{X/S} \arrow[r, "\varepsilon"] & \mathcal{O}_X \arrow[r] & 0
\end{tikzcd}$$

where $J \subset \Delta^*(\mathcal{O}_{X\times_S X})$ is the kernel of $\Delta^\#_{X/S}$. We again view $\mathcal{P}^1_{X/S}$ and $\Omega^1_{X/S}$ as $\mathcal{O}_X$-modules via the map induced by $p_1 : X\times_S X \to X$. If we factor $\Delta_{X/S}$ as

$$X \xleftarrow{i} U \xrightarrow{j} X\times_S X$$

with i a closed immersion and j an open immersion, then $J = i^*(I)$ where $I \subset \mathcal{O}_U$ is the quasicoherent ideal that defines i. It follows that $\Omega^1_{X/S} = J/J^2$ is a quasicoherent $\mathcal{O}_X$-module. We claim that also $\mathcal{P}^1_{X/S}$ is quasicoherent. Indeed, we have the following general result.

<u>Lemma</u> Let $(X, \mathcal{O}_X)$ be a scheme and let

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of $\mathcal{O}_X$-modules. If two out of three of M', M, and M'' are quasicoherent, then so is the third.

Pf We proved in lecture 7 that an $\mathcal{O}_X$-module N is quasicoherent if and only if for all $U \subset V \subset X$ affine open, the canonical map

$$\Gamma(U, \mathcal{O}_X) \otimes_{\Gamma(V, \mathcal{O}_X)} \Gamma(V, N) \to \Gamma(U, N)$$

is an isomorphism. But $\Gamma(U, \mathcal{O}_X)$ is a flat $\Gamma(V, \mathcal{O}_X)$-module, since open immersions are flat. (Flatness can checked stalkwise, and for all $x \in U$, $\Gamma(U, \mathcal{O}_X)_{\mathfrak{p}_x} = \mathcal{O}_{X,x} = \Gamma(V, \mathcal{O}_X)_{\mathfrak{p}_x}$.) The statement follows from the obvious five-lemma argument. //

It follows that if $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$ are affine, then

$$0 \to \Omega^1_{X/S} \to \mathcal{P}^1_{X/S} \xrightarrow{\varepsilon} \mathcal{O}_X \to 0$$

is canonically isomorphic to the sequence of quasicoherent $\mathcal{O}_X$-modules obtained from the sequence of A-modules

$$0 \to \Omega^1_{A/R} \to P^1_{A/R} \xrightarrow{\varepsilon} A \to 0.$$

Let again $f: X \to S$ be any morphism of schemes, and let M be an $\mathcal{O}_X$-module, not necessary coherent. An $f^*\mathcal{O}_S$-linear map

$$\mathcal{O}_X \xrightarrow{D} M$$

is an S-derivation from $\mathcal{O}_X$ to M if for every open $V \subset X$ and all sections $t_1, t_2 \in \Gamma(V, \mathcal{O}_X)$,

$$D(t_1 t_2) = t_1 D(t_2) + t_2 D(t_1).$$

Here, that D is $f^*\mathcal{O}_S$-linear means that for every open $V \subset X$, every open $U \subset S$ s.t. $V \subset f^{-1}(U)$, and all sections $s \in \Gamma(U, \mathcal{O}_S)$ and $t \in \Gamma(V, \mathcal{O}_X)$,

$$D(f^\#(s)|_V \cdot t) = f^\#(s)|_V \cdot D(t).$$

We write $\mathrm{Der}_S(\mathcal{O}_X, M)$ for the

$\Gamma(X, \mathcal{O}_X)$-module of S-derivations from $\mathcal{O}_X$ to M. If $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$ are affine and $M = \tilde{N}$ quasicoherent, then the map

$$\mathrm{Der}_S(\mathcal{O}_X, M) \longrightarrow \mathrm{Der}_R(A, N)$$

that takes D to the induced map of global sections is an isomorphism of A-modules.

For any $f: X \to S$, we have two sections

$$\begin{tikzcd} \mathcal{P}^1_{X/S} \arrow[r, "\varepsilon"] & \mathcal{O}_X \arrow[l, bend right, "d^0"'] \arrow[l, bend left, "d^1"] \end{tikzcd}$$

induced from the two projections $p_1, p_2: X \times_S X \to X$. We consider $\mathcal{P}^1_{X/S}$ an $\mathcal{O}_X$-module via d^0. The difference $d^0 - d^1$ factors

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "d"] \arrow[rr, bend right, "d^0 - d^1"'] & \Omega^1_{X/S} \arrow[r] & \mathcal{P}^1_{X/S} \end{tikzcd}$$

and we see as earlier that d is an S-derivation from $\mathcal{O}_X$ to $\Omega^1_{X/S}$.

Prop Let $f: X \to S$ be a morphism of schemes and let M be an $\mathcal{O}_X$-module. The $\Gamma(X, \mathcal{O}_X)$-linear map

$$\mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/S}, M) \longrightarrow \mathrm{Der}_S(\mathcal{O}_X, M)$$
$$h \longmapsto h \circ d$$

is an isomorphism.

Pf Again, the map is injective, since $\Omega^1_{X/S}$ is generated locally by the image of d. So let $D: \mathcal{O}_X \to M$ be an S-derivation. For all affine open $V \subset X$ and $U \subset S$ s.t. $f(U) \subset V$, $D_V: \Gamma(V, \mathcal{O}_X) \to \Gamma(V, M)$ is a $\Gamma(U, S)$-linear derivation. Hence, there exists a unique $\Gamma(V, \mathcal{O}_X)$-linear map $u_V: \Gamma(V, \Omega^1_{X/S}) \to \Gamma(V, M)$ s.t. $D_V = u_V \circ d$. By the uniqueness of the u_V, these glue to give a map of $\mathcal{O}_X$-modules $u: \mathcal{O}_X \to M$ s.t. $D = u \circ d$. //

We proceed to define differential operators, $\mathrm{Hom}_{\mathcal{O}_X}(P^1_{X/S}, M)$ being the $\mathcal{O}_X$-module of differential operators of degree ≤ 1.

We fix a morphism of schemes

$$X \xrightarrow{f} S$$

and define a Hopf algebroid in the category of $f^*\mathcal{O}_S$-algebras in $X^\sim$. A Hopf algebroid is the data that corepresents a groupoid. In the case at hand, the set of objects is corepresented by $\mathcal{O}_X$, the set of morphisms by

$$\mathcal{P}_{X/S} = \Delta^*_{X/S}\mathcal{O}_{X\times_S X} = \mathcal{O}_X \otimes_{f^*\mathcal{O}_S} \mathcal{O}_X ,$$

the target and source maps by

$$\mathcal{O}_X \xrightarrow{d^0} \mathcal{P}_{X/S}, \qquad \mathcal{O}_X \xrightarrow{d^1} \mathcal{P}_{X/S}$$

$$a \longmapsto a\otimes 1 \qquad\qquad a \longmapsto 1\otimes a$$

respectively, the composition by

$$\mathcal{P}_{X/S} \xrightarrow{\delta} \mathcal{P}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S}$$

$$a\otimes b \longmapsto a\otimes 1 \otimes_{\mathcal{O}_X} 1\otimes b ,$$

the identity morphism by

$$\begin{aligned} \mathcal{P}_{X/S} &\xrightarrow{\ \varepsilon\ } \mathcal{O}_X \\ a\otimes b &\longmapsto ab \end{aligned},$$

and the inverse of morphisms by

$$\begin{aligned} \mathcal{P}_{X/S} &\xrightarrow{\ \sigma\ } \mathcal{P}_{X/S} \\ a\otimes b &\longmapsto b\otimes a \end{aligned}$$

Let M, N be $\mathcal{O}_X$-modules. An $f^*\mathcal{O}_S$-linear map

$$M \xrightarrow{\ h\ } N$$

determines and is determined by the $\mathcal{O}_X$-linear map

$$\begin{aligned} \mathcal{P}_{X/S} \otimes_{\mathcal{O}_X} M &\xrightarrow{\ \bar{h}\ } N \\ a\otimes b \otimes_{\mathcal{O}_X} x &\longmapsto a\,h(bx) \end{aligned}$$

Moreover, the map h is $\mathcal{O}_X$-linear if and only if the map $\bar{h}$ annihilates $J \otimes_{\mathcal{O}_X} M$. We weaken the latter condition as follows. For $n \geq 0$, define

$$\mathcal{P}^n_{X/S} = \mathcal{P}_{X/S} / J^{n+1}.$$

The Hopf algebroid structure maps

induce maps

$$\mathcal{O}_X \underset{d^1_n}{\overset{d^0_n}{\rightrightarrows}} \mathcal{P}^n_{X/S}, \qquad \mathcal{P}^n_{X/S} \xrightarrow{\varepsilon_n} \mathcal{O}_X$$

$$\mathcal{P}^{m+n}_{X/S} \xrightarrow{\delta_{m,n}} \mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{P}^m_{X/S}$$

$$\mathcal{P}^n_{X/S} \xrightarrow{\sigma_n} \mathcal{P}^n_{X/S}$$

Let again M, N be two $\mathcal{O}_X$-modules. We define a <u>differential operator</u> of <u>degree</u> $\leq n$ from M to N ~~to~~ be an $\mathcal{O}_X$-linear map

$$\mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} M \longrightarrow N$$

and write

$$\mathrm{Diff}^n_{X/S}(M, N)$$

$$= \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} M, N)$$

for the set of differential operators of degree $\leq n$ from M to N.

<u>Rmk</u> I should have given the definition of the tensor product of two $\mathcal{O}_X$-modules M, N earlier.

It is defined to be the sheaf associated with the presheaf

$$U \longmapsto \Gamma(U,M) \underset{\Gamma(U,\mathcal{O}_X)}{\otimes} \Gamma(U,N).$$

The functor $M \otimes_{\mathcal{O}_X} -$ has a right adjoint $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(M,-)$ defined by

$$\Gamma(U, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(M,N))$$

$$= \mathrm{Hom}_{\mathcal{O}_X|_U}(M|_U, N|_U).$$

It is an exercise to show that $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(M,N)$ is indeed a sheaf. //

We define the sheaf of differential operators from M to N of degree $\leq n$ by

$$\underline{\mathrm{Diff}}^n_{X/S}(M,N)$$

$$= \underline{\mathrm{Hom}}_{\mathcal{O}_X}(P^n_{X/S} \otimes_{\mathcal{O}_X} M, N).$$

It is a $P_{X/S}$-module, hence, an $\mathcal{O}_X$-$\mathcal{O}_X$-bimodule. The map

$$\mathcal{O}_X \xrightarrow{d^0_n} P^n_{X/S}$$

induces a (forgetful) map

$$\underline{\mathrm{Diff}}^n_{X/S}(M,N) \longrightarrow \mathcal{H}om_{f^*\mathcal{O}_S}(M,N).$$

This map is injective, because the image of d^0_n locally generates $\mathcal{P}^n_{X/S}$ as an $\mathcal{O}_X$-module (w.r.t. the map d^1_n). Hence, we may view differential operators of degree $\leq n$ from M to N as $f^*\mathcal{O}_S$-linear maps from M to N that satisfy a condition weaker than $\mathcal{O}_X$-linearity. There are more general situations (e.g. PD-differential operators) in which the corresponding map is not injective, so it is better to leave the definition of the sheaf $\underline{\mathrm{Diff}}^n_{X/S}(M,N)$ as it is.

The canonical epimorphism

$$\mathcal{P}^m_{X/S} \twoheadrightarrow \mathcal{P}^n_{X/S} \qquad m \geq n$$

induces a monomorphism

$$\underline{\mathrm{Diff}}^n_{X/S}(M,N) \hookrightarrow \underline{\mathrm{Diff}}^m_{X/S}(M,N),$$

and we define the sheaf of

differential operators (of all degr-ees) from M to N to be the (filtered) colimit

$$\underline{\mathrm{Diff}}_{X/S}(M,N)$$

$$= \operatorname*{colim}_{n} \underline{\mathrm{Diff}}^{n}_{X/S}(M,N);$$

it is an $\mathcal{O}_X$-$\mathcal{O}_X$-bimodule. If M, N, and P are three $\mathcal{O}_X$-modules, the the composition map $\circ_{m,n}$ gives rise to a map of $\mathcal{O}_X$-$\mathcal{O}_X$-bimodules

$$\underline{\mathrm{Diff}}^{n}_{X/S}(N,P) \otimes_{\mathcal{O}_X} \underline{\mathrm{Diff}}^{m}_{X/S}(M,N)$$
$$\xrightarrow{\circ} \underline{\mathrm{Diff}}^{m+n}_{X/S}(M,P)$$

which, in turn, induces a map of $\mathcal{O}_X$-$\mathcal{O}_X$-bimodules

$$\underline{\mathrm{Diff}}_{X/S}(N,P) \otimes_{\mathcal{O}_X} \underline{\mathrm{Diff}}_{X/S}(M,N)$$
$$\xrightarrow{\circ} \underline{\mathrm{Diff}}_{X/S}(M,P).$$

<u>Ex</u> Let S be a scheme, let

$$X = S \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}[x_1,\dots,x_d]),$$

and let $f = \mathrm{pr}_1 : X \to S$. For $1 \leq i \leq d$, we set

$$\xi_i = 1\otimes x_i - x_i \otimes 1 \in \Gamma(X, \mathcal{P}_{X/S})$$

and denote by the same symbol the image in $\Gamma(X, \mathcal{P}^n_{X/S})$. For all $n \geq 0$, we have:

(i) $\mathcal{P}^n_{X/S}$ is a free $\mathcal{O}_X$-module with the basis

$$\{ \xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d} \mid \alpha_1 + \cdots + \alpha_d \leq n \}$$

where $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d_{\geq 0}$.

(ii) If $\{ D_\alpha \mid \alpha_1 + \cdots + \alpha_d \leq n \}$ be the dual basis of $\mathrm{Diff}^n_{X/S}(\mathcal{O}_X, \mathcal{O}_X)$, then

$$D_\alpha \circ D_{\alpha'} = \binom{\alpha+\alpha'}{\alpha} D_{\alpha+\alpha'}$$

where

$$\alpha+\alpha' = (\alpha_1+\alpha'_1, \ldots, \alpha_d+\alpha'_d)$$

$$\binom{\beta}{\alpha} = \binom{\beta_1}{\alpha_1} \cdots \binom{\beta_d}{\alpha_d}. \qquad //$$