Problems for Recitation 2

The purpose this week is to prove (5) below. This result is know by the French term *recollement*, which means something like reattachment.

Let X be a space, let $U \subset X$ be an open subset, and let $Y \subset X$ be the closed complement. We write $i\colon Y \to X$ and $j\colon U \to X$ for the canonical inclusions.

(1) Let $j^{-1}\colon O(X) \to O(U)$ be the inverse image functor and recall the adjoint pair of functors (j^p, j_p) with $j_p = (j^{-1})^*\colon U^\wedge \to X^\wedge$ and with $j^p = (j^{-1})_!\colon X^\wedge \to U^\wedge$ the left Kan extension. Show that j^p and j_p both preserve sheaves.

(2) We consider the functor $u\colon O(U) \to O(X)$ defined by $u(V \subset U) = V \subset X$, let $u^*\colon X^\wedge \to U^\wedge$ be the induced functor, and let $u_!, u_*\colon U^\wedge \to X^\wedge$ be the left and right Kan extensions, respectively. Show that u^* and j^p (resp. u_* and j_p) are canonically isomorphic. Conclude that the functor

$$j_! = a_X\, u_!\, i_U\colon U^\sim \to X^\sim$$

is left adjoint to j^*.

(3) Prove that the functor $u_!\colon U^\wedge \to X^\wedge$ from (2) is given by

$$(u_!F)(V) = \begin{cases} F(V) & \text{if } V \subset U \\ \emptyset & \text{if } V \not\subset U. \end{cases}$$

(Here $\emptyset$ is the initial object in the category of sets. If we were considering presheaves in some other category, we would get an initial object in that category instead.)

(4) We let F be a sheaf on X and consider the following diagram of sheaves on X in which the horizontal maps (resp. the vertical maps) are induced from the unit map of the adjoint pair (j^*, j_*) (resp. of the adjoint pair (i^*, i_*)).

$$\begin{array}{ccc} F & \longrightarrow & j_*j^*F \\ \downarrow & & \downarrow \\ i_*i^*F & \longrightarrow & i_*i^*j_*j^*F \end{array}$$

Show that the diagram is cartesian. (*Hint*: Calculate the induced diagrams of stalks at $x \in X$, considering the two cases $x \in U$ and $x \in Y$ separately.)

(5) Define $(Y^\sim, U^\sim, i^*j_*)$ to be the category, where the objects are triples (F_1, F_2, f) with F_1 and F_2 sheaves on Y and U, respectively, and with $f\colon F_1 \to i^*j_*F_2$ a map of sheaves on Y, and where a morphism from (F_1, F_2, f) to (F_1', F_2', f') is a pair (g_1, g_2) of a morphisms $g_1\colon F_1 \to F_1'$ and $g_2\colon F_2 \to F_2'$ such that the diagram

$$\begin{array}{ccc} F_1 & \xrightarrow{\;f\;} & i^*j_*F_2 \\ \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle i^*j_*g_2} \\ F_1' & \xrightarrow{\;f'\;} & i^*j_*F_2' \end{array}$$

commutes. Conclude from (4) that the functor

$$X^\sim \to (Y^\sim, U^\sim, i^*j_*)$$

that takes F to $(i^*F, j^*F, i^*\eta \colon i^*F \to i^*j_*j^*F)$ is an equivalence of categories.

(6) Show that, under the equivalence in (5), the functors i^*, i_*, $j_!$, j^*, and j_* are given by the following formulas.

$$\begin{aligned}
i^*(F_1, F_2, f) &= F_1 \\
i_*(F_1) &= (F_1, *, F_1 \to *) \\
j_!(F_2) &= (\emptyset, F_2, \emptyset \to F_2) \\
j^*(F_1, F_2, f) &= F_2 \\
j_*(F_2) &= (i^*j_*F_2, F_2, \mathrm{id})
\end{aligned}$$