Problems for Recitation 3

1. (i) Show that the functor from schemes to sets that to $(X, \mathcal{O}_X)$ associates the underlying set of the ring $\Gamma(X, \mathcal{O}_X)$ is representable by the affine line

$$\mathbb{A}^1_{\mathbb{Z}} = (\operatorname{Spec}(\mathbb{Z}[t]), \mathcal{O}_{\operatorname{Spec}(\mathbb{Z}[t])}).$$

In other words, show that there is a natural bijection

$$\operatorname{Hom}_{\mathrm{Sch}}((X, \mathcal{O}_X), \mathbb{A}^1_{\mathbb{Z}}) \xrightarrow{h_X} \Gamma(X, \mathcal{O}_X).$$

(ii) Conclude from Yoneda's lemma that the affine line $\mathbb{A}^1_{\mathbb{Z}}$ has a structure of ring object in the category of schemes which, via the natural bijection h, gives rise to the ring structure on the set $\Gamma(X, \mathcal{O}_X)$.

(iii) Determine the ring homomorphisms

$$\mathbb{Z}[t] \underset{\epsilon_1}{\overset{\epsilon_0}{\rightrightarrows}} \mathbb{Z}, \qquad \mathbb{Z}[t] \underset{\Delta^{\times}}{\overset{\Delta^{+}}{\rightrightarrows}} \mathbb{Z}[t] \otimes \mathbb{Z}[t]$$

that define the ring object structure on $\mathbb{A}^1_{\mathbb{Z}}$.

2. Let $S = \bigoplus_{d \geqslant 0} S_d$ be a graded ring and let $S_+ \subset S$ be the homogeneous ideal of elements of positive degree. (Note that 0 has all degrees, while 1 has degree 0.) Let $f \in S_+$ be a homogeneous element of degree d. The ring of fractions $S_f = S[\frac{1}{f}]$ is naturally $\mathbb{Z}$-graded, and we define

$$S_{(f)} = \{\frac{a}{f^n} \in S_f \mid a \in S_{nd}\} \subset S_f$$

to be the subring of elements of degree 0 and $X_f = (\operatorname{Spec} S_{(f)}, \mathcal{O}_{\operatorname{Spec} S_{(f)}})$.

(i) Let $f, g \in S_+$ be homogeneous elements. Show that the morphism

$$\varphi_{fg} \colon X_{fg} \to X_f, \qquad \frac{ag^n}{(fg)^n} \leftarrow\!\shortmid \frac{a}{f^n},$$

is an open immersion. (*Hint*: The image is a distinguished open.)

(ii) Show that the schemes X_f glue to give a scheme

$$X = (\operatorname{Proj}(S), \mathcal{O}_{\operatorname{Proj}(S)}).$$

(This scheme, being a colimit, is well-defined, up to unique isomorphism.)

(iii) Let $D_+(f) \subset X$ denote the image of the canonical map $\mathrm{in}_f \colon X_f \to X$. Show that as f varies over the set of homogeneous elements of S_+, the open subsets $D_+(f) \subset X$ form a basis for the topology on X.

(iv) Show that $\{D_+(f_i)\}_{i \in I}$ cover X if and only if for all $x \in S_+$, some power of x is contained in the ideal generated by the elements f_i, $i \in I$. (*Hint*: Note that $\{D_+(f_i)\}_{i \in I}$ cover X if and only if each $D_+(f)$ is covered by $\{D_+(f) \cap D_+(f_i)\}_{i \in I}$.)