Problems for Recitation 4

The purpose of this problem set is to construct the normalization of an integral scheme. We accomplish this in a series of steps.

1. Let X be an irreducible topological space. Show that every non-empty open subset $U \subset X$ is dense and that U itself is an irreducible topological space. (A space X is irreducible if whenever $X = V_1 \cup V_2$ with $V_1, V_2 \subset X$ closed, then either $V_1 = X$ or $V_2 = X$ or both.)

2. A scheme X is *integral* if for every open subset $U \subset X$, the ring $\Gamma(U, \mathcal{O}_X)$ is an integral domain. Show that a scheme is integral if and only if it is reduced and irreducible. (A scheme X is reduced if for every open subset $U \subset X$, the ring $\Gamma(U, \mathcal{O}_X)$ does not contain any nilpotent elements.)

3. Let $f\colon X \to X'$ be a morphism between integral schemes. Show that the following are equivalent:

(i) the image $f(X) \subset X'$ is dense;

(ii) if $U \subset X$ and $U' \subset X'$ are affine open subsets such that $f(U) \subset U'$, then the composite ring homomorphism

$$\Gamma(U', \mathcal{O}_{X'}) \to \Gamma(f^{-1}(U'), \mathcal{O}_{X'}) \to \Gamma(U, \mathcal{O}_X)$$

is injective.

Such a map f is called *dominant*. (*Hint*: to show that (i) implies (ii), show that if f is in the kernel, then for all $x' \in U'$, $f(x') = 0$. Conclude that $f = 0$. To show that (ii) implies (i), show that f maps the generic point of U to the generic point of U'.)

4. An integral scheme X is *normal* if for every affine open subset $U \subset X$, the ring $\Gamma(U, \mathcal{O}_X)$ is integrally closed (in its quotient field). Let X be an integral scheme. A dominant morphism $f\colon \tilde{X} \to X$ with $\tilde{X}$ normal is called a *normalization* of X if it is universal with this property, i.e. if every dominant morphism $g\colon Z \to X$ with Z normal factors uniquely through f.

(i) Suppose that $X = \operatorname{Spec} R$ is affine, and let $R \to \tilde{R}$ be the canonical map from R to the integral closure of the ring R in its quotient field. Show that the induced map $\operatorname{Spec} \tilde{R} \to \operatorname{Spec} R$ is a normalization.

(ii) Suppose that $\tilde{X} \to X$ is a normalization and let $U \subset X$ be an open subset. Show that the projection $U \times_X \tilde{X} \to U$ is a normalization. (*Hint:* Use the universal properties.)

(iii) Show that every integral scheme X has a normalization $\tilde{X} \to X$. (*Hint:* Let $\{U_i\}$ be an affine open cover of X; use (i) to find a normalization $\tilde{U}_i \to U_i$; and use (ii) to show that the schemes $\tilde{U}_i$ can be glued together to give $\tilde{X}$. Show that the maps $\tilde{U}_i \to U_i$ glue to give $\tilde{X} \to X$.)