Problems for Recitation 5

A morphism of schemes $f\colon X \to Y$ is *quasi-compact* if for every affine open subset $V \subset Y$, the inverse image $f^{-1}(V) \subset X$ is quasi-compact. If X is a topological space, if $x \in X$, and if x' is an element of the closure of $\{x\} \subset X$, then we say that $x' \in X$ is a *specialization* of x.

1. Show that for a quasi-compact morphism of schemes $f\colon X \to Y$, the following are equivalent.

(i) The morphism f is closed.

(ii) For every $x \in X$ and for every specialization y' of $y = f(x) \in Y$, there exists a specialization x' of x such that $f(x') = y'$.

Let K be a field. A *valuation ring* in K is a subring $V \subset K$ that is not a field and such that for all $a \in K^*$, either $a \in V$ or $a^{-1} \in V$ or both. The valuation rings in K are maximal among the local rings $A \subset K$ that are not fields with respect to domination. (If $A, B \subset K$ are two local rings, then B dominates A if $A \subset B$ and $\mathfrak{m}_A = A \cap \mathfrak{m}_B$.) The prime spectrum $\operatorname{Spec}(V)$ of a valuation ring V has a unique closed point s and a unique generic point η and s is a specialization of η. (The elements of $\operatorname{Spec}(V)$ are in one-to-one correspondence with the convex subgroups of the value group K^*/V^*, which is totally ordered under the inclusion relation $aV^* \subset bV^*$.) We write $j\colon \operatorname{Spec}(K) \to \operatorname{Spec}(V)$ for the morphism of schemes induced by the inclusion of V in K.

2. Let $f\colon X \to S$ be a morphism of schemes, let $x \in X$, let $y = f(x) \in Y$, and let $y' \neq y$ be a specialization of y. Show that there exists a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}(K) & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle f} \\ \operatorname{Spec}(V) & \xrightarrow{g} & S \end{array}$$

with K a field and V a valuation ring in K such that $g(s) = y'$ and $g'(\eta) = x$.

3. Suppose that the diagonal morphism $\Delta_{X/S}\colon X \to X \times_S X$ is quasi-compact. Show that if, in every diagram of the form

$$\begin{array}{ccc} \operatorname{Spec}(K) & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle f} \\ \operatorname{Spec}(V) & \xrightarrow{g} & S \end{array}$$

with K a field and V a valuation ring in K, at most one morphism $h\colon \operatorname{Spec}(V) \to X$ such that $h \circ j = g'$ and $f \circ h = g$ exists, then f is separated.

4. Let $f\colon X \to S$ be a quasi-compact and separated morphism, and suppose that, in every diagram as in Problem 3, a (necessarily unique) morphism $h\colon \operatorname{Spec}(V) \to X$ with $h \circ j = g'$ and $f \circ h = g$ exists. Show that f is universally closed.

(*Hint*: First show that f is closed. Then show that the hypothesis is satisfied for every base change of the morphism f.)