Problems for Recitation 7

The problems this time concern affine morphisms of schemes. A morphism of schemes $f\colon X \to S$ is defined to be affine if for every affine open subset $V \subset S$, the inverse image $f^{-1}(V) \subset X$ is an affine open subset. The problems show that the functor that to $f\colon X \to S$ assigns $f_*(\mathcal{O}_X)$ is an equivalence of categories from the category of affine S-schemes to the category of quasicoherent $\mathcal{O}_S$-algebras.

1. Let $f\colon X \to S$ be a morphism of schemes and suppose that S admits a covering $\{V_i\}_{i \in I}$ by affine open subsets with the property that for every $i \in I$, the inverse image $f^{-1}(V_i) \subset X$ is affine open. Show that $f\colon X \to S$ is an affine morphism.

2. Show that if $f\colon X \to S$ is an affine morphism, then the $\mathcal{O}_S$-algebra $f_*(\mathcal{O}_X)$ is quasicoherent.

3. Let S be a scheme and let $\mathcal{A}$ be a quasicoherent $\mathcal{O}_S$-algebra. Prove the following statements:

(i) If $U \subset V \subset S$ are affine open, then the diagram of canonical maps

$$
\begin{array}{ccc}
\mathrm{Spec}(\Gamma(U, \mathcal{A})) & \longrightarrow & \mathrm{Spec}(\Gamma(V, \mathcal{A})) \\
\downarrow & & \downarrow \\
U & \longrightarrow & V
\end{array}
$$

is cartesian. Here the left-hand vertical map is the composition

$$\mathrm{Spec}(\Gamma(U, \mathcal{A})) \longrightarrow \mathrm{Spec}(\Gamma(U, \mathcal{O}_S)) \xrightarrow{\ \epsilon\ } U$$

of the map induced by the unit of the $\mathcal{O}_S$-algebra $\mathcal{A}$ and the counit of the adjunction $(\Gamma(-,-), \mathrm{Spec})$, and the right-hand vertical map is analogous.

(ii) There exists, up to canonical isomorphism of schemes over S, a unique morphism $f\colon X \to S$ such that $f_*(\mathcal{O}_X)$ is isomorphic to $\mathcal{A}$.

(iii) Show that this morphism $f\colon X \to A$ is affine.

We write $f\colon \mathrm{Spec}_S(\mathcal{A}) \to S$ for the morphism constructed in (ii).

4. Let S be a fixed scheme, and let Aff_S and $\mathrm{Alg}_{\mathcal{O}_S,\mathrm{qc}}$ be the categories of affine S-schemes and quasicoherent $\mathcal{O}_S$-algebras, respectively. Prove that

$$\mathrm{Aff}_S \underset{\mathrm{Spec}_S}{\overset{f_*}{\rightleftarrows}} \mathrm{Alg}^{\mathrm{op}}_{\mathcal{O}_S,\mathrm{qc}}$$

is an adjoint equivalence of categories.