Problems for Recitation 8

1. Let B be an A-algebra of finite presentation, let $h\colon A[x_1,\dots,x_n] \to B$ be a surjective A-algebra homomorphism, and let $I \subset A[x_1,\dots,x_n]$ be the kernel of h. Show that the ideal $I \subset A[x_1,\dots,x_n]$ necessarily is finitely generated.

2. Show that if $f\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is étale, then B admits a presentation

$$B = A[x_1,\dots,x_n]/(f_1,\dots,f_n)$$

such that the image in $M_n(B)$ of the Jacobian $(\partial f_i/\partial x_j)$ is invertible.

3. Let $f\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ be an étale morphism between affine schemes and suppose that $A = \operatorname{colim}_i A_i$ be a filtered colimit of rings. Show that there exists a cartesian square

$$\begin{array}{ccc} \operatorname{Spec}(B) & \longrightarrow & \operatorname{Spec}(B_i) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_i} \\ \operatorname{Spec}(A) & \longrightarrow & \operatorname{Spec}(A_i) \end{array}$$

with f_i étale and with the lower horizontal map induced by the canonical ring homomorphism $\mathrm{in}_i\colon A_i \to A$.

4. Show that if $f\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is a smooth morphism between affine schemes, then there exists a cartesian square

$$\begin{array}{ccc} \operatorname{Spec}(B) & \longrightarrow & \operatorname{Spec}(B_0) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_0} \\ \operatorname{Spec}(A) & \longrightarrow & \operatorname{Spec}(A_0) \end{array}$$

with f_0 smooth and with the lower horizontal map induced by the canonical inclusion of a subring $A_0 \subset A$ that is of finite type over $\mathbb{Z}$.