Problems for Recitation 1

1. Let C be a category and let $h\colon C \to C\hat{}$ be the Yoneda embedding. Show that the map that to a sieve S on X assigns the subfunctor $h(S)$ of $h(X)$ defined by

$$h(S)(Y) = \{f\colon Y \to X \mid f \in \mathrm{ob}(S)\} \subset h(X)(Y)$$

is a bijection from the set of sieves on X to the set of subfunctors of $h(X)$. Show that if $g\colon X' \to X$ is a morphism in C and S a sieve on X, then the diagram

$$\begin{array}{ccc} h(g^*(S)) & \longrightarrow & h(S) \\ \downarrow & & \downarrow \\ h(X') & \xrightarrow{h(g)} & h(X) \end{array}$$

is a fiber product in $\mathsf{C}\hat{}$. Here the vertical maps are the canonical inclusions and the top horizontal map is the natural transformation whose value at Y' is the map that to $f'\colon Y' \to X'$ to $g \circ f'\colon Y' \to X$. Restate the definitions of a topology J on C and of a sheaf on C with respect to the topology J in terms of subfunctors of representable presheaves.

2. Let C be a category which admits fiber products. If K is a pretopology on C, then we define the topology J_K on C generated by K as follows. For every object X of C, the set $J_K(X)$ consists of all sieves S on X with the property that there exists a family $(f_i\colon X_i \to X)_{i \in I}$ in $K(X)$ all of whose members are objects of S. Show that J_K is indeed a topology. Show that a presheaf F on C is a sheaf with respect to K if and only if it is a sheaf with respect to J_K.