Problems for Recitation 3

1. A continuous map of spaces $f\colon X \to Y$ gives rise to a functor $u\colon O(Y) \to O(X)$ defined by $u(U) = f^{-1}(U)$. Show that u is continuous, i.e., that $(u^{op})^*$ preserves sheaves (see Example 3.5 in the lecture notes). In this setting the functor u_s is denoted by f_* and called the direct image while u^s is denoted by f^* and called the inverse image functor. Describe the effects of f_* and f^* on sheaves.

2. Let X be a topological space and let $x \in X$ be a point. Given a set S, we define the skyscraper sheaf of S at x by

$$\mathrm{skysc}_x(S)(U) = \begin{cases} S & \text{if } x \in U \\ \{\emptyset\} & \text{if } x \notin U, \end{cases}$$

with either identity maps or the unique map to $\{\emptyset\}$ as restriction maps. Show that $\mathrm{skysc}_x(S)$ is a sheaf and that the construction gives a functor

$$\mathrm{skysc}_x\colon \mathit{Set} \to X^{\sim}$$

to the category $X^{\sim} = O(X)^{\sim}$ of sheaves on X.

Let F be a sheaf on X. We define the stalk of F at $x \in X$ to be the colimit

$$\mathrm{stalk}_x(F) = \mathrm{colim}_{x \in U} F(U)$$

indexed the category of open sets containing x. Show that this construction defines a functor

$$\mathrm{stalk}_x\colon X^{\sim} \to \mathit{Set}.$$

Give an alternative description of skysc_x and stalk_x in terms of continuous maps of spaces and show that they are adjoint functors.

3. A morphism $f\colon B \to C$ in a category C is a *monomorphism* if for all pairs of morphisms $g, h\colon A \to B$ the equality $f \circ g = f \circ h$ implies the equality $g = h$. Similarly, f is an *epimorphism* if for all pairs of morphisms $g, h\colon C \to D$, the equality $g \circ f = h \circ f$ implies the equality $g = h$.

(a) Show that in the category *Set* the monomorphisms are exactly the injective functions and the epimorphisms are exactly the surjective functions. Show that in the category *Ring* of rings and ring homomorphisms the monomorphisms are injective but that not all epimorphisms are surjective.

(b) Show that $f\colon B \to C$ is a monomorphism if and only if the diagram

$$\begin{array}{ccc} B & \xrightarrow{id} & B \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{f} & C \end{array}$$

is cartesian. Use this to show that right adjoints preserve monomorphisms. State and prove the corresponding result for epimorphisms and left adjoints.

(c) Let (C, J) be a site. Show that a map $f\colon F \to G$ of presheaves is a monomorphism or epimorphism if and only if for every object X in C, the map $f_X\colon F(X) \to G(X)$ is a monomorphism or epimorphism, respectively. Show that a map $f\colon F \to$

G of sheaves is a monomorphism if and only it is a monomorphism as a map of presheaves.

Epimorphisms of sheaves have a more complicated description:

Proposition. *Let $f\colon F \to G$ be a map of sheaves on (C, J). Then f is an epimorphism if and only if for each object X in C and each $x \in G(X)$, there is a covering sieve $S \in J(X)$ such that for all $g\colon Y \to X \in \mathrm{ob}(S)$, the element $F(g)(x)$ is in the image of $f_Y : F(Y) \to G(Y)$.*

(d) Show that a map of sheaves is an isomorphism if and only if it is both a monomorphism and an epimorphism.

4. Let $\mathbb{C}$ be the set of complex numbers with the usual topology. Let $\mathcal{O}_{\mathbb{C}}^{\mathrm{an}}$ be the sheaf of holomorphic functions on $\mathbb{C}$ defined by

$$\mathcal{O}_{\mathbb{C}}^{\mathrm{an}}(U) = \{f\colon U \to \mathbb{C} \mid f \text{ is holomorphic on } U\}.$$

Define $\mathcal{O}_{\mathbb{C}}^{\mathrm{an}\,*}$ to be the subsheaf of $\mathcal{O}_{\mathbb{C}}^{\mathrm{an}}$ of nowhere vanishing holomorphic functions. The exponential map is the map of sheaves

$$\exp\colon \mathcal{O}_{\mathbb{C}}^{\mathrm{an}} \to \mathcal{O}_{\mathbb{C}}^{\mathrm{an}*}$$

given on sections by $\exp(f)(z) = e^{f(z)}$. Show that exp is an epimorphism of sheaves but that it is not surjective on all open sets.