Problems for Recitation 4

1. Let U be a universe and let C be a locally U-small and U-cocomplete abelian category in which U-small filtered colimits and finite limits commute. Assume, in addition, that C has an object G such that the functor

$$\mathsf{C} \xrightarrow{\mathsf{C}(G,-)} U\text{-}\mathsf{Set}$$

is faithful. Prove the following statements.

(i) If $j\colon A'' \to A'$ is a monomorphism in C and if the induced map

$$\mathsf{C}(G,A'') \xrightarrow{\mathsf{C}(G,j)} \mathsf{C}(G,A')$$

is a bijection, then j is an isomorphism.

(ii) If $i\colon A' \to A$ and $i'\colon A'' \to A$ are two monomorphisms in C and if the images of the induced maps

$$\mathsf{C}(G,A') \xrightarrow{\mathsf{C}(G,i')} \mathsf{C}(G,A) \qquad \text{and} \qquad \mathsf{C}(G,A'') \xrightarrow{\mathsf{C}(G,i)} \mathsf{C}(G,A)$$

are equal, then there exists an isomorphism $j\colon A'' \to A'$ such that $i' = i \circ j$.

(iii) Conclude that for every object A in C, the set of subobjects of A is bijective to a U-small set.

2. Let C be a category, and let $h_{\mathsf{C}}\colon \mathsf{C} \to \mathsf{C}^{\wedge}$ be the Yoneda embedding. Let X be an object of $\mathsf{C}^{\wedge}$, let h_{C}/X be the slice category, and let $j_X\colon h_{\mathsf{C}}/X \to \mathsf{C}$ be the forgetful functor.

(i) Let J be a topology on C. Show that there is a topology J/X on h_{C}/X defined be declaring a sieve $S \subset h_{h_{\mathsf{C}}/X}(c,u)$ to be a covering sieve if and only $(j_X^{\mathrm{op}})_!(S) \subset h(c)$ is a covering sieve on c.

(ii) Show that $j_X\colon h_{\mathsf{C}}/X \to \mathsf{C}$ is both continuous and cocontinuous.

(iii) Show that the functor $j_{X!}\colon (h_{\mathsf{C}}/X, J/X)^{\sim} \to (\mathsf{C}, J)^{\sim}$ factors as

$$(h_{\mathsf{C}}/X, J/X)^{\sim} \xrightarrow{(e_X)^{\sim}} (\mathsf{C}, J)^{\sim}/a(X) \longrightarrow (\mathsf{C}, J)^{\sim},$$

where the right-hand functor is the forgetful functor, and that the left-hand functor is an equivalence of categories. Here $a\colon \mathsf{C}^{\wedge} \to \mathsf{C}^{\sim}$ is the sheafification functor.

(iv) Conclude that if E is a topos and X and object of E, then E/X is a topos and the forgetful functor $E/X \to E$ a morphism of topoi.