Problems for Recitation 5

1. Let U be a universe, and let U-Sch be the category of U-schemes. Show that a U-Set valued presheaf F on U-Sch is a sheaf for the fpqc-topology if and only if it satifies the following (i)–(ii):

(i) F is a sheaf for the Zariski topology.
(ii) For every faithfully flat morphism $f\colon W \to V$ between affine U-schemes,

$$F(V) \xrightarrow{F(f)} F(W) \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} F(W \times_V W)$$

is an equalizer diagram in U-Set.

2. Let U be a universe, let $u\colon \mathsf{C} \to \mathsf{C}'$ be a functor from a U-small category C to a locally U-small category C', and let J' be a topology on C'.

(i) Show that there is a finest topology J among the topologies on C for which the functor $u\colon (\mathsf{C}, J) \to (\mathsf{C}', J')$ is continuous.
(ii) Show that if C and C' admit finite limits and u preserves finite limits, then the sieve on X generated by the family $(f_i\colon X_i \to X)_{i\in I}$ is in $J(X)$ if and only if the sieve generated by the family $(u(f_i)\colon u(X_i) \to u(X))_{i\in I}$ is in $J'(u(X))$.

The topology J is called the induced topology on C relative to the topology J' on C' and the functor $u\colon \mathsf{C} \to \mathsf{C}'$.

3. Let again U be a universe and let U-Sch be the category of U-schemes. We define the *multiplicative group* to be the abelian group object $\mathbb{G}_m$ in the category of U-Set valued presheaves on U-Sch that to a U-scheme X assigns the abelian group $\mathbb{G}_m(X) = \Gamma(X, \mathscr{O}_X)^*$. Find a representation of $\mathbb{G}_m$ by an abelian group object in U-Sch and conclude that $\mathbb{G}_m$ is a sheaf for the fpqc-topology.