Report problems[1]

Problem 1. *Due*: Tuesday, May 1, 2018, in Science Building 1, Room 105.

Let X be the prime spectrum of a ring and let $x \in X$.

(i) Show that the closure $\{x\}^-$ of the one-point subset $\{x\} \subset X$ is an irreducible closed subset in the sense that it cannot be written as the union of two proper closed subsets.
(ii) Show that x is a generic point of $\{x\}^-$ in the sense that the only closed subset of $\{x\}^-$ which contains x is the whole set.
(iii) Show that every irreducible closed subset of X is of the form $\{x\}^-$ and that x is its unique generic point. Conclude that the assignment $x \mapsto \{x\}^-$ defines a one-to-one correspondance between the points of X and the irreducible closed subsets of X.

Problem 2. *Due*: Tuesday, May 8, 2018, in Science Building 1, Room 105.

Let C be a category. We consider diagrams $\mathcal{X}\colon I \to \mathsf{C}$ indexed by various index categories I and the limits of such diagrams, if they exists. If I is discrete in the sense that every morphism in I is an identity morphism, then a diagram $\mathcal{X}\colon I \to \mathsf{C}$ determines and is determined by the family $(\mathcal{X}(i))_{i\in \mathrm{ob}(I)}$ of objects in C. In this situation, a limit of the diagram $\mathcal{X}\colon I \to \mathsf{C}$ is said to be a *product* of the family $(\mathcal{X}(i))_{i\in \mathrm{ob}(I)}$ and is denoted by

$$\prod_{i\in \mathrm{ob}(I)} \mathcal{X}(i).$$

If I is the category with two objects 0 and 1 and with two parallel morphisms $f, g\colon 0 \to 1$ (in addition to the identity morphisms of 0 and 1), then a diagram $\mathcal{X}\colon I \to \mathsf{C}$ determines and is determined by the two parallel morphisms

$$\mathcal{X}(0) \underset{\mathcal{X}(g)}{\overset{\mathcal{X}(f)}{\rightrightarrows}} \mathcal{X}(1).$$

In this situation, a limit of the diagram $\mathcal{X}\colon I \to \mathsf{C}$ is said to be an *equalizer* of the parallel morphisms $\mathcal{X}(f)$ and $\mathcal{X}(g)$.

Now, we let $\mathcal{X}\colon I \to \mathsf{C}$ be any diagram in C and assume that the products

$$\prod_{i\in \mathrm{ob}(I)} \mathcal{X}(i) \qquad \text{and} \qquad \prod_{f\colon i\to j\in \mathrm{mor}(I)} \mathcal{X}(j)$$

indexed by the set $\mathrm{ob}(I)$ of objects in I and the set $\mathrm{mor}(I)$ of morphisms in I, respectively, both exist. We consider the unique morphisms

$$\prod_{i\in \mathrm{ob}(I)} \mathcal{X}(i) \underset{b}{\overset{a}{\rightrightarrows}} \prod_{f\colon i\to j\in \mathrm{mor}(I)} \mathcal{X}(j)$$

such that for every morphism $f\colon i \to j$ in $\mathrm{mor}(I)$,

$$p_{f\colon i\to j} \circ a = \mathcal{X}(f) \circ p_i \qquad \text{and} \qquad p_{f\colon i\to j} \circ b = p_j.$$

[1] Course homepage: www.math.nagoya-u.ac.jp/~larsh/teaching/S2018_A

Prove the following statements:

(i) If $(p_i\colon X \to \mathcal{X}(i))_{i\in \mathrm{ob}(I)}$ is a limit of $\mathcal{X}\colon I \to \mathsf{C}$, then

$$X \xrightarrow{(p_i)} \prod_{i\in \mathrm{ob}(I)} \mathcal{X}(i)$$

is an equalizer of a and b.

(ii) Conversely, if $(p_i)\colon X \to \prod_{i\in \mathrm{ob}(I)} \mathcal{X}(i)$ is an equalizer of a and b, then

$$(p_i\colon X \to \mathcal{X}(i))_{i\in \mathrm{ob}(I)}$$

is a limit of $\mathcal{X}\colon I \to \mathsf{C}$.

Problem 3. *Due*: Tuesday, May 22, 2018, in Science Building 1, Room 105.

We have defined an *adjunction* from a category C to a category D to be a triple of a functor $F\colon \mathsf{C} \to \mathsf{D}$, a functor $G\colon \mathsf{D} \to \mathsf{C}$, and for every pair (c,d) of an object in C and an object in D, a bijection

$$\mathrm{Hom}_{\mathsf{D}}(F(c),d) \xrightarrow{\alpha_{(c,d)}} \mathrm{Hom}_{\mathsf{C}}(c,G(d)).$$

which is natural in c and d. That the bijection be natural means that if $f\colon c_0 \to c_1$ is a morphism in C and d is an object in D, then the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{D}}(F(c_1),d) & \xrightarrow{\alpha_{(c_1,d)}} & \mathrm{Hom}_{\mathsf{C}}(c,G(d)) \\ \big\downarrow{\scriptstyle \mathrm{Hom}_{\mathsf{D}}(F(f),d)} & & \big\downarrow{\scriptstyle \mathrm{Hom}_{\mathsf{C}}(f,G(d))} \\ \mathrm{Hom}_{\mathsf{D}}(F(c_0),d) & \xrightarrow{\alpha_{(c_0,d)}} & \mathrm{Hom}_{\mathsf{C}}(c,G(d)) \end{array}$$

commutes, and, similarly, if c is an object in C and $g\colon d_0 \to d_1$ is a morphism in D, then the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{D}}(F(c),d_0) & \xrightarrow{\alpha_{(c,d_0)}} & \mathrm{Hom}_{\mathsf{C}}(c,G(d_0)) \\ \big\downarrow{\scriptstyle \mathrm{Hom}_{\mathsf{D}}(c,g)} & & \big\downarrow{\scriptstyle \mathrm{Hom}_{\mathsf{C}}(c,G(g))} \\ \mathrm{Hom}_{\mathsf{D}}(F(c),d_1) & \xrightarrow{\alpha_{(c,d_1)}} & \mathrm{Hom}_{\mathsf{C}}(c,G(d_1)) \end{array}$$

commutes. In particular, for every object c in C, we have the morphism

$$\eta_c = a_{(c,F(c))}(\mathrm{id}_{F(c)})\colon c \to G(F(c)),$$

and the family of morphisms $(\eta_c)_{c\in\mathrm{ob}(\mathsf{C})}$ constitute a natural transformation

$$\mathrm{id}_{\mathsf{C}} \xRightarrow{\eta} G\circ F$$

of functors from C to C. Similarly, for every object d of D, we have the morphism

$$\epsilon_d = a^{-1}_{(G(d),d)}(\mathrm{id}_{G(d)})\colon F(G(d)) \to d,$$

and the family of morphisms $(\eta_d)_{d\in\mathrm{ob}(\mathsf{D})}$ constitute a natural transformation

$$F\circ G \xRightarrow{\epsilon} \mathrm{id}_{\mathsf{D}}$$

of functors from D to D. Prove the following statements:

(i) The diagrams of natural transformations

$$\begin{array}{ccccccc} F & \xRightarrow{F\circ\eta} & F\circ G\circ F & \qquad & G & \xRightarrow{\eta\circ G} & G\circ F\circ G \\ & \searrow\!\!\!\!= & \swarrow_{\epsilon\circ F} & & & \searrow\!\!\!\!= & \swarrow_{G\circ\epsilon} \\ & F & & & & G & \end{array}$$

commute. That the left-hand diagram commutes means that for every object c in C, the following diagram of morphisms in C commutes.

$$\begin{array}{ccc} F(c) & \xrightarrow{F(\eta_c)} & F(G(F(c))) \\ & \searrow\!\!\!\!= \quad \swarrow_{\epsilon_{F(c)}} & \\ & F(c) & \end{array}$$

The natural transformations $\epsilon\colon F\circ G \Rightarrow \mathrm{id}_{\mathsf{D}}$ and $\eta\colon \mathrm{id}_{\mathsf{C}} \Rightarrow G\circ F$ are called the *counit* and the *unit* of the adjunction, respectively.

(ii) Let $\epsilon\colon F\circ G \Rightarrow \mathrm{id}_{\mathsf{D}}$ and $\eta\colon \mathrm{id}_{\mathsf{C}} \Rightarrow G\circ F$ be natural transformations such that the triangular diagrams in (i) commute. Given objects c in C and d in D, we define

$$\mathrm{Hom}_{\mathsf{D}}(F(c), d) \xrightarrow{\alpha_{(c,d)}} \mathrm{Hom}_{\mathsf{C}}(c, G(d))$$

to be the composite map

$$\mathrm{Hom}_{\mathsf{D}}(F(c), d) \xrightarrow{G} \mathrm{Hom}_{\mathsf{C}}(G(F(c)), d) \xrightarrow{\mathrm{Hom}_{\mathsf{C}}(\eta_c, d)} \mathrm{Hom}_{\mathsf{C}}(c, G(d)).$$

Show that $\alpha_{(c,d)}$ is a bijection (what is the inverse?) and that it is natural in c and d. Conclude that (F, G, α) is an adjunction.

The problem shows that we may define an adjunction from C to D, equivalently, to be a quadruple (F, G, ϵ, η) of functors $F\colon \mathsf{C}\to\mathsf{D}$ and $G\colon \mathsf{D}\to\mathsf{C}$ and natural transformations $\epsilon\colon F\circ G \Rightarrow \mathrm{id}_{\mathsf{D}}$ and $\eta\colon \mathrm{id}_{\mathsf{C}} \Rightarrow G\circ F$ that make the triangular diagrams in (i) commute.

Problem 4. *Due*: Tuesday, June 12, 2018, in Science Building 1, Room 105.

The purpose this problem is to prove (v) below. This result is know by the French term *recollement*, which means something like reattachment. We let X be a space, let $U \subset X$ be an open subset, and let $Y \subset X$ be the closed complement. We write $i\colon Y \to X$ and $j\colon U \to X$ for the canonical inclusions.

(i) Let $j^{-1}\colon O(X) \to O(U)$ be the inverse image functor and recall the adjoint pair of functors (j^p, j_p) with $j_p = (j^{-1})^*\colon U^\wedge \to X^\wedge$ and with $j^p = (j^{-1})_!\colon X^\wedge \to U^\wedge$ the left Kan extension. Show that the functors j^p and j_p both preserve sheaves.

(ii) We consider the functor $u\colon O(U) \to O(X)$ that to $V \subset U$ assigns $V \subset X$, let $u^*\colon X^\wedge \to U^\wedge$ be the induced functor, and let $u_!, u_*\colon U^\wedge \to X^\wedge$ be the left and right Kan extensions, respectively. Show that u^* and j^p (resp. u_* and j_p) are canonically naturally isomorphic, and conclude that the functor

$$j_! = a_X\, u_!\, i_U\colon U^\sim \to X^\sim$$

is left adjoint to j^*.

(iii) Prove that the functor $u_! \colon U^\wedge \to X^\wedge$ from (2) is given by

$$(u_! F)(V) = \begin{cases} F(V) & \text{if } V \subset U \\ \emptyset & \text{if } V \not\subset U. \end{cases}$$

(Here $\emptyset$ is the initial object in the category of sets. If we were considering presheaves in some other category, we would get an initial object in that category instead.)

(iv) We let F be a sheaf on X and consider the following diagram of sheaves on X in which the horizontal maps (resp. the vertical maps) are induced from the unit map of the adjoint pair (j^*, j_*) (resp. of the adjoint pair (i^*, i_*)).

$$\begin{array}{ccc} F & \longrightarrow & j_* j^* F \\ \downarrow & & \downarrow \\ i_* i^* F & \longrightarrow & i_* i^* j_* j^* F \end{array}$$

Show that the diagram is a pullback.[2] [*Hint*: Calculate the induced diagrams of stalks at $x \in X$, considering the cases $x \in U$ and $x \in Y$ separately.]

(v) Define $(Y^\sim, U^\sim, i^* j_*)$ to be the category, where the objects are triples (F_1, F_2, f) with F_1 and F_2 sheaves on Y and U, respectively, and with $f \colon F_1 \to i^* j_* F_2$ a map of sheaves on Y, and where a morphism from (F_1, F_2, f) to (F_1', F_2', f') is a pair (g_1, g_2) of a morphisms $g_1 \colon F_1 \to F_1'$ and $g_2 \colon F_2 \to F_2'$ such that the diagram

$$\begin{array}{ccc} F_1 & \xrightarrow{f} & i^* j_* F_2 \\ \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle i^* j_* g_2} \\ F_1' & \xrightarrow{f'} & i^* j_* F_2' \end{array}$$

commutes. Conclude from (iv) that the functor

$$X^\sim \to (Y^\sim, U^\sim, i^* j_*)$$

that takes F to $(i^* F, j^* F, i^* \eta \colon i^* F \to i^* j_* j^* F)$ is an equivalence of categories.

(vi) Show that, under the equivalence in (v), the functors i^*, i_*, $j_!$, j^*, and j_* are given by the following formulas.

$$\begin{aligned} i^*(F_1, F_2, f) &= F_1 \\ i_*(F_1) &= (F_1, *, F_1 \to *) \\ j_!(F_2) &= (\emptyset, F_2, \emptyset \to F_2) \\ j^*(F_1, F_2, f) &= F_2 \\ j_*(F_2) &= (i^* j_* F_2, F_2, \mathrm{id}) \end{aligned}$$

[2] You may use the following theorem without proof: A morphism $f \colon F \to G$ in $X^\sim$ is an isomorphisms if and only if for every $x \in X$, the induced map of stalks $f_x \colon F_x \to G_x$ is a bijection.

Problem 5. *Due*: Tuesday, July 3, 2018, in Science Building 1, Room 105.

A morphism of schemes $f\colon X \to Y$ is *quasi-compact* if for every affine open subset $V \subset Y$, the inverse image $f^{-1}(V) \subset X$ is quasi-compact. If X is a topological space, if $x \in X$, and if x' is an element of the closure of $\{x\} \subset X$, then we say that $x' \in X$ is a *specialization* of x.

(i) Show that if $f\colon X \to Y$ is a quasi-compact morphism of schemes, then the following are equivalent.
   (a) The morphism f is closed. (This means that the underlying map of topological spaces is a closed map.)
   (b) For every $x \in X$ and for every specialization y' of $y = f(x) \in Y$, there exists a specialization x' of x such that $f(x') = y'$.

Let K be a field. A *valuation ring* in K is a subring $V \subset K$ that is not a field and such that for all $a \in K^*$, either $a \in V$ or $a^{-1} \in V$ or both. The valuation rings in K are maximal among the local rings $A \subset K$ that are not fields with respect to domination. (If $A, B \subset K$ are two local rings, then B dominates A if $A \subset B$ and $\mathfrak{m}_A = A \cap \mathfrak{m}_B$.) The prime spectrum $\operatorname{Spec}(V)$ of a valuation ring V has a unique closed point s and a unique generic point η and s is a specialization of η. (The elements of $\operatorname{Spec}(V)$ are in one-to-one correspondence with the convex subgroups of the value group K^*/V^*, which is totally ordered under the inclusion relation $aV^* \subset bV^*$.) We write $j\colon \operatorname{Spec}(K) \to \operatorname{Spec}(V)$ for the morphism of schemes induced by the inclusion of V in K.

(ii) Let $f\colon X \to S$ be a morphism of schemes, let $x \in X$, let $y = f(x) \in Y$, and let $y' \neq y$ be a specialization of y. Show that there exists a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}(K) & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle f} \\ \operatorname{Spec}(V) & \xrightarrow{g} & S \end{array}$$

with K a field and V a valuation ring in K such that $g(s) = y'$ and $g'(\eta) = x$.