## **Report** problems<sup>1</sup>

Problem 1. Due: Tuesday, May 1, 2018, in Science Building 1, Room 105.

Let X be the prime spectrum of a ring and let  $x \in X$ .

- (i) Show that the closure  $\{x\}^-$  of the one-point subset  $\{x\} \subset X$  is an irreducible closed subset in the sense that it cannot be written as the union of two proper closed subsets.
- (ii) Show that x is a generic point of  $\{x\}^-$  in the sense that the only closed subset of  $\{x\}^-$  which contains x is the whole set.
- (iii) Show that every irreducible closed subset of X is of the form  $\{x\}^-$  and that x is its unique generic point. Conclude that the assignment  $x \mapsto \{x\}^-$  defines a one-to-one correspondance between the points of X and the irreducible closed subsets of X.

Problem 2. Due: Tuesday, May 8, 2018, in Science Building 1, Room 105.

Let C be a category. We consider diagrams  $\mathcal{X}: I \to \mathsf{C}$  indexed by various index categories I and the limits of such diagrams, if they exists. If I is discrete in the sense that every morphism in I is an identity morphism, then a diagram  $\mathcal{X}: I \to \mathsf{C}$ determines and is determined by the family  $(\mathcal{X}(i))_{i \in ob(I)}$  of objects in  $\mathsf{C}$ . In this situation, a limit of the diagram  $\mathcal{X}: I \to \mathsf{C}$  is said to be a *product* of the family  $(\mathcal{X}(i))_{i \in ob(I)}$  and is denoted by

$$\prod_{i\in \mathrm{ob}(I)}\mathcal{X}(i).$$

If I is the category with two objects 0 and 1 and with two parallel morphisms  $f, g: 0 \to 1$  (in addition to the identity morphisms of 0 and 1), then a diagram  $\mathcal{X}: I \to \mathsf{C}$  determines and is determined by the two parallel morphisms

$$\mathcal{X}(0) \xrightarrow[\mathcal{X}(g)]{\mathcal{X}(g)} \mathcal{X}(1).$$

In this situation, a limit of the diagram  $\mathcal{X} \colon I \to \mathsf{C}$  is said to be an *equalizer* of the parallel morphisms  $\mathcal{X}(f)$  and  $\mathcal{X}(g)$ .

Now, we let  $\mathcal{X}: I \to \mathsf{C}$  be any diagram in  $\mathsf{C}$  and assume that the products

$$\prod_{i \in \mathrm{ob}(I)} \mathcal{X}(i) \qquad \text{and} \qquad \prod_{f: i \to j \in \mathrm{mor}(I)} \mathcal{X}(j)$$

indexed by the set ob(I) of objects in I and the set mor(I) of morphisms in I, respectively, both exist. We consider the unique morphisms

$$\prod_{i \in ob(I)} \mathcal{X}(i) \xrightarrow[b]{a} \prod_{f: i \to j \in mor(I)} \mathcal{X}(j)$$

such that for every morphism  $f: i \to j$  in mor(I),

$$p_{f:i \to j} \circ a = \mathcal{X}(f) \circ p_i$$
 and  $p_{f:i \to j} \circ b = p_j$ .

 $<sup>^1\,{\</sup>rm Course}$  homepage: www.math.nagoya-u.ac.jp/~larsh/teaching/S2018\_A

Prove the following statements:

(i) If  $(p_i: X \to \mathcal{X}(i))_{i \in ob(I)}$  is a limit of  $\mathcal{X}: I \to \mathsf{C}$ , then

$$X \xrightarrow{(p_i)} \prod_{i \in \mathrm{ob}(I)} \mathcal{X}(i)$$

is an equalizer of a and b.

(ii) Conversely, if  $(p_i): X \to \prod_{i \in ob(I)} \mathcal{X}(i)$  is an equalizer of a and b, then  $(p_i: X \to \mathcal{X}(i))_{i \in ob(I)}$ 

is a limit of  $\mathcal{X}: I \to \mathsf{C}$ .

Problem 3. Due: Tuesday, May 22, 2018, in Science Building 1, Room 105.

We have defined an *adjunction* from a category C to a category D to be a triple of a functor  $F: C \to D$ , a functor  $G: D \to C$ , and for every pair (c, d) of an object in C and an object in D, a bijection

$$\operatorname{Hom}_{\mathsf{D}}(F(c),d) \xrightarrow{\alpha_{(c,d)}} \operatorname{Hom}_{\mathsf{C}}(c,G(d)).$$

which is natural in c and d. That the bijection be natural means that if  $f: c_0 \to c_1$  is a morphism in C and d is an object in D, then the diagram

$$\operatorname{Hom}_{\mathsf{D}}(F(c_{1}),d) \xrightarrow{\alpha_{(c_{1},d)}} \operatorname{Hom}_{\mathsf{C}}(c,G(d))$$

$$\downarrow^{\operatorname{Hom}_{\mathsf{D}}(F(f),d)} \qquad \qquad \downarrow^{\operatorname{Hom}_{\mathsf{C}}(f,G(d))}$$

$$\operatorname{Hom}_{\mathsf{D}}(F(c_{0}),d) \xrightarrow{\alpha_{(c_{0},d)}} \operatorname{Hom}_{\mathsf{C}}(c,G(d))$$

commutes, and, similarly, if c is an object in C and  $g: d_0 \to d_1$  is a morphism in D, then the diagram

commutes. In particular, for every object c in C, we have the morphism

$$\eta_c = a_{(c,F(c))}(\mathrm{id}_{F(c)}) \colon c \to G(F(c)),$$

and the family of morphisms  $(\eta_c)_{c \in ob(\mathsf{C})}$  constitute a natural transformation

$$\operatorname{id}_{\mathsf{C}} \xrightarrow{\eta} G \circ F$$

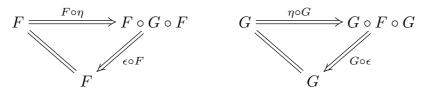
of functors from C to C. Similarly, for every object d of D, we have the morphism  $\epsilon_d = a_{(G(d),d)}^{-1}(\mathrm{id}_{G(d)}) \colon F(G(d)) \to d,$ 

and the family of morphisms  $(\eta_d)_{d \in ob(D)}$  constitute a natural transformation

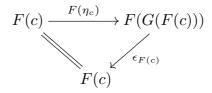
$$F \circ G \xrightarrow{\epsilon} \operatorname{id}_{\mathsf{D}}$$

of functors from D to D. Prove the following statements:

(i) The diagrams of natural transformations



commute. That the left-hand diagram commutes means that for every object c in C, the following diagram of morphisms in C commutes.



The natural transformations  $\epsilon \colon F \circ G \Rightarrow \mathrm{id}_{\mathsf{D}}$  and  $\eta \colon \mathrm{id}_{\mathsf{C}} \Rightarrow G \circ F$  are called the *counit* and the *unit* of the adjunction, respectively.

(ii) Let  $\epsilon: F \circ G \Rightarrow \operatorname{id}_{\mathsf{D}}$  and  $\eta: \operatorname{id}_{\mathsf{C}} \Rightarrow G \circ F$  be natural transformations such that the triangular diagrams in (i) commute. Given objects c in  $\mathsf{C}$  and d in  $\mathsf{D}$ , we define

$$\operatorname{Hom}_{\mathsf{D}}(F(c),d) \xrightarrow{\alpha_{(c,d)}} \operatorname{Hom}_{\mathsf{C}}(c,G(d))$$

to be the composite map

$$\operatorname{Hom}_{\mathsf{D}}(F(c),d) \xrightarrow{G} \operatorname{Hom}_{\mathsf{C}}(G(F(c)),d) \xrightarrow{\operatorname{Hom}_{\mathsf{C}}(\eta_c,d)} \operatorname{Hom}_{\mathsf{C}}(c,G(d)).$$

Show that  $\alpha_{(c,d)}$  is a bijection (what is the inverse?) and that it is natural in c and d. Conclude that  $(F, G, \alpha)$  is an adjunction.

The problem shows that we may define an adjunction from C to D, equivalently, to be a quadruple  $(F, G, \epsilon, \eta)$  of functors  $F: C \to D$  and  $G: D \to C$  and natural transformations  $\epsilon: F \circ G \Rightarrow id_{D}$  and  $\eta: id_{C} \Rightarrow G \circ F$  that make the triangular diagrams in (i) commute.

Problem 4. Due: Tuesday, June 12, 2018, in Science Building 1, Room 105.

The purpose this problem is to prove (v) below. This result is know by the French term *recollement*, which means something like reattachment. We let X be a space, let  $U \subset X$  be an open subset, and let  $Y \subset X$  be the closed complement. We write  $i: Y \to X$  and  $j: U \to X$  for the canonical inclusions.

- (i) Let  $j^{-1}: O(X) \to O(U)$  be the inverse image functor and recall the adjoint pair of functors  $(j^p, j_p)$  with  $j_p = (j^{-1})^* : U^{\wedge} \to X^{\wedge}$  and with  $j^p = (j^{-1})_! : X^{\wedge} \to U^{\wedge}$  the left Kan extension. Show that the functors  $j^p$  and  $j_p$  both preserve sheaves.
- (ii) We consider the functor  $u: O(U) \to O(X)$  that to  $V \subset U$  assigns  $V \subset X$ , let  $u^*: X^{\wedge} \to U^{\wedge}$  be the induced functor, and let  $u_!, u_*: U^{\wedge} \to X^{\wedge}$  be the left and right Kan extensions, respectively. Show that  $u^*$  and  $j^p$  (resp.  $u_*$ and  $j_p$ ) are canonically naturally isomorphic, and conclude that the functor

$$j_! = a_X \, u_! \, i_U \colon U^\sim \to X^\sim$$

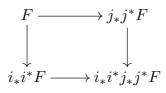
is left adjoint to  $j^*$ .

(iii) Prove that the functor  $u_!: U^{\wedge} \to X^{\wedge}$  from (2) is given by

$$(u_!F)(V) = \begin{cases} F(V) & \text{if } V \subset U \\ \emptyset & \text{if } V \not\subset U. \end{cases}$$

(Here  $\emptyset$  is the initial object in the category of sets. If we were considering presheaves in some other category, we would get an initial object in that category instead.)

(iv) We let F be a sheaf on X and consider the following diagram of sheaves on X in which the horizontal maps (resp. the vertical maps) are induced from the unit map of the adjoint pair  $(j^*, j_*)$  (resp. of the adjoint pair  $(i^*, i_*)$ ).



Show that the diagram is a pullback.<sup>2</sup> [*Hint*: Calculate the induced diagrams of stalks at  $x \in X$ , considering the cases  $x \in U$  and  $x \in Y$  separately.] (v) Define  $(Y^{\sim}, U^{\sim}, i^*j_*)$  to be the category, where the objects are triples

 $(F_1, F_2, f)$  with  $F_1$  and  $F_2$  sheaves on Y and U, respectively, and with  $f: F_1 \to i^* j_* F_2$  a map of sheaves on Y, and where a morphism from  $(F_1, F_2, f)$  to  $(F'_1, F'_2, f')$  is a pair  $(g_1, g_2)$  of a morphisms  $g_1: F_1 \to F'_1$  and  $g_2: F_2 \to F'_2$  such that the diagram

$$\begin{array}{c} F_1 & \stackrel{f}{\longrightarrow} i^* j_* F_2 \\ \downarrow^{g_1} & \downarrow^{i^* j_* g_2} \\ F'_1 & \stackrel{f'}{\longrightarrow} i^* j_* F'_2 \end{array}$$

commutes. Conclude from (iv) that the functor

$$X^{\sim} \to (Y^{\sim}, U^{\sim}, i^* j_*)$$

that takes F to  $(i^*F, j^*F, i^*\eta: i^*F \to i^*j_*j^*F)$  is an equivalence of categories.

(vi) Show that, under the equivalence in (v), the functors  $i^*$ ,  $i_*$ ,  $j_!$ ,  $j^*$ , and  $j_*$  are given by the following formulas.

$$i^{*}(F_{1}, F_{2}, f) = F_{1}$$

$$i_{*}(F_{1}) = (F_{1}, *, F_{1} \to *)$$

$$j_{!}(F_{2}) = (\emptyset, F_{2}, \emptyset \to F_{2})$$

$$j^{*}(F_{1}, F_{2}, f) = F_{2}$$

$$j_{*}(F_{2}) = (i^{*}j_{*}F_{2}, F_{2}, id)$$

<sup>&</sup>lt;sup>2</sup> You may use the following theorem without proof: A morphism  $f: F \to G$  in  $X^{\sim}$  is an isomorphisms if and only if for every  $x \in X$ , the induced map of stalks  $f_x: F_x \to G_x$  is a bijection.

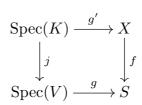
Problem 5. Due: Tuesday, July 3, 2018, in Science Building 1, Room 105.

A morphism of schemes  $f: X \to Y$  is quasi-compact if for every affine open subset  $V \subset Y$ , the inverse image  $f^{-1}(V) \subset X$  is quasi-compact. If X is a topological space, if  $x \in X$ , and if x' is an element of the closure of  $\{x\} \subset X$ , then we say that  $x' \in X$  is a specialization of x.

- (i) Show that if  $f: X \to Y$  is a quasi-compact morphism of schemes, then the following are equivalent.
  - (a) The morphism f is closed. (This means that the underlying map of topological spaces is a closed map.)
  - (b) For every  $x \in X$  and for every specialization y' of  $y = f(x) \in Y$ , there exists a specialization x' of x such that f(x') = y'.

Let K be a field. A valuation ring in K is a subring  $V \subset K$  that is not a field and such that for all  $a \in K^*$ , either  $a \in V$  or  $a^{-1} \in V$  or both. The valuation rings in K are maximal among the local rings  $A \subset K$  that are not fields with respect to domination. (If  $A, B \subset K$  are two local rings, then B dominates A if  $A \subset B$  and  $\mathfrak{m}_A = A \cap \mathfrak{m}_B$ .) The prime spectrum  $\operatorname{Spec}(V)$  of a valuation ring V has a unique closed point s and a unique generic point  $\eta$  and s is a specialization of  $\eta$ . (The elements of  $\operatorname{Spec}(V)$  are in one-to-one correspondence with the convex subgroups of the value group  $K^*/V^*$ , which is totally ordered under the inclusion relation  $aV^* \subset bV^*$ .) We write j:  $\operatorname{Spec}(K) \to \operatorname{Spec}(V)$  for the morphism of schemes induced by the inclusion of V in K.

(ii) Let  $f: X \to S$  be a morphism of schemes, let  $x \in X$ , let  $y = f(x) \in Y$ , and let  $y' \neq y$  be a specialization of y. Show that there exists a commutative diagram



with K a field and V a valuation ring in K such that g(s) = y' and  $g'(\eta) = x$ .